

**Advanced Math: Notes on Lessons 9-12 (plus notes on Lessons 1-8)**  
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**Overall discussion of class.**

- Can't teach everything (no time) – more like a tutor for most difficult/most critical concepts.
- Schedule: Each week has 4 lessons (I suggest that you do them Monday..Thursday), plus a test of the previous week's lessons (I suggest doing the test on Friday). So, you'll have a chance to ask questions during class *before* you take a test on it. Classes will cover “this Tuesday through the following Monday material”, so the class will always cover material you are *about* to do.
- May need to adjust as we go; please let me know of issues early, so we can adjust before they become real problems. In particular, I'm presuming a level of knowledge/starting point that may be wrong; if so, please help me understand where you *really* are.
- Will need to work...! ~hour/day + class. Do the problems so you understand the material. The point is *understanding*, including being *able to apply it...* not rote work. You can just do the odd-numbered or even-numbered problems, if you like. I won't look at your practice problems, so if really understand the material, you don't need to do the problems at all. But it would be unwise to not do the practice problems. It's very easy to read the lesson and think you understand the material. Doing the practice problems will help you really understand and remember the material.

**Motivation – Why Math? (Separate discussion & handout)**

**Lesson 9:**

Congruent: Easy; just means it can be turned/moved to exactly overlap. IE: same shape and size.

Know the basic triangle congruency postulates: SSS, SAS, AAAS (ASA), HL. *Key building blocks*; in geometry we often reduce constructs to triangles, and then prove that the triangles are congruent.

Proof outlines: Proofs are *creative*; it's hard to teach how to create new proofs! Hints for getting started:

- Draw/write out “what you know” (e.g., picture, draw the tick marks on lengths and angles)
- State or at least note “obvious” inferences from that (using, e.g., SSS, SAS, AAAS, HL, proportionality, angles you can calculate by having 2 triangle angles / supplementary angles (total  $90^\circ$ ) / complementary angles (total  $180^\circ$ )) – and do that repeatedly if easy to do
- Look at what you're trying to prove – how can you get there from what you have? Look for steps going forwards (from what you know) or backwards (from what you want) that might connect “what you know” with “what you want”. Break things down (e.g., break shapes into triangles).

**Lesson 10:**

Equation of a line (for non-vertical line):  $y = mx + b$

- $m$  = slope, the amount up/down every 1 unit right
- $b$  = y-intercept; when  $x=0$ , what's  $y$ ?

So line with  $y = 3x + 2$  means that its slope is 3 (3 up for every one right), y-intercept ( $x=0$ ) is 2.

Perpendicular line's slope  $m_2 = -(1/m)$

Square root of 2 is an irrational number (can't exactly express as a fraction). This discovery is typically credited to Hippasus (aka Hippasos) of Metapontum, born ca. 500 B.C. Before this, the Pythagoreans preached that all numbers could be expressed as the ratio of integers. Reportedly the Pythagoreans either exiled or murdered Hippasus (accounts differ). More generally, any counting number root of a counting number is either a counting number or an irrational number. Pi is irrational too.

Completing the square / quadratic equation (lesson 11):

If you see an equation of this form:

$$ax^2 + bx + c = 0$$

How can you solve for x?!? This is *really hard* if you don't know how. It's also a very common form of equation, for example,  $ax^2 + bx + c$  can calculate the (one-dimensional) position over time of an object that is accelerating, such as by gravity (if  $a$  = acceleration such as the acceleration from gravity,  $b$  = starting velocity,  $c$  = starting position, and  $x$  is time, then  $ax^2 + bx + c$  describes an object's position over time; the quadratic equation will let you determine when an object reaches a certain position, such as the ground). Gravity is rather common!

Thankfully, this puzzle has been solved, and it's so important that you need to memorize its solution; not only because the problem itself is important, but because it's a good example of how to solve harder problems (change its form into something you *can* solve, then use that). To understand it, it's probably easier to look at going the "the other way" (which is easier & gives a clue on how to do it).

E.G.:

$$(x+3)^2 = 0 \quad (\text{This is easy, } x = -3)$$

$$x^2 + 6x + 9 = 0 \quad (\text{Multiplied out})$$

Notice the "3" is doubled to a "6". So if you saw exactly  $x^2 + 6x + 9$ , you could replace it with  $(x+3)^2$ . In general, if the constant is the square of half the "x" term, and  $x^2$  has a multiplier of 1, you could replace it with the square. "Completing the square" is just a math trick to turn existing equations into a format that you *can* handle. Here it is:

<b>Result</b>	<b>Rationale</b>
$ax^2 + bx + c = 0, a \neq 0$	Given
$x^2 + (b/a)x + c/a = 0$	Divide the equation by "a", so that x doesn't have anything in front. Note that $0/a = 0$ .
$x^2 + (b/a)x + b^2/(4a^2) - b^2/(4a^2) + c/a = 0$	The big trick! We know $x^2 + (b/a)x$ could become $(x + b/(2a))^2$ if we had $x + (b/a)x + b^2/(4a^2)$ . The big trick is, let's create that format by adding $0 = b^2/(4a^2) - b^2/(4a^2)$ . Adding 0 won't change anything! (You could add $b^2/(4a^2)$ to both side instead, if you like.)
$(x + b/(2a))^2 - b^2/(4a^2) + c/a = 0$	Now the first part is a square, so replace it.
$(x + b/(2a))^2 = b^2/(4a^2) - c/a$	Simplify - move the rest to the other side.
$(x + b/(2a))^2 = b^2/(4a^2) - 4ac/(4a^2)$	Simplify - multiply $(c/a)$ by $4a/4a$ so its denominator is the same as the other part
$(x + b/(2a))^2 = (b^2 - 4ac) / (4a^2)$	Combine fraction, since denominator is the same.
$x + \frac{b}{2a} = \pm \sqrt{(b^2 - 4ac) / (4a^2)}$	Take the square root of both sides. Note: square roots may have a + and - result. We were given $a \neq 0$ .
$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \quad (a \neq 0)$	Simplify down to the quadratic equation.

Memorize the quadratic equation, and memorize how to derive it (by completing the square). You need to be able to re-do this, both as equation and as proof.

**Lesson 11:**

Defines many terms and properties about circles, and the quadratic formula. (No need to duplicate them here, they're pretty clear.) Memorize, memorize! For example:

1. circle's intercepted arc (degrees of the arc of the circle's edge) = 2 x inscribed angle (angle starting elsewhere on the circle)
2. if two chords of a circle intersect, the products of chord 1's segments = product of chord 2's segments
3. two tangent segments from a point outside a circle have equal lengths

**Lesson 12:**

Sum of exterior angles of a polygon is  $360^\circ$  (go full circle)

Sum of interior angles of a convex polygon N sides =  $(N-2)180^\circ = (\# \text{ triangles})(\text{degrees in a triangle})$

Chord+tangent that intersect at tangent's point of contact form an angle on a circle's edge; its angle is  $\frac{1}{2}$  the measure of the intercepted arc.

**Previous material (Lessons 1-8):**

Some important issues include:

**Lesson 1:**

Review the definitions. Remember that complementary angles sum to  $90^\circ$ , supplementary angles sum to  $180^\circ$ . Note the rules for equal angles on transversals, and definitions of triangles (esp. isosceles triangles).

**Lesson 2:**

Memorize how to calculate areas and volumes of common shapes, particularly:

Area of Rectangle = (base) (height) = bh

Area of Triangle =  $(\frac{1}{2})$  (base) (height) =  $(\frac{1}{2})$  bh

Area of circle =  $\pi r^2$ ; Circumference of circle =  $2\pi r$

Volume of Cylinders (inc. Prisms) = (area of base) (perpendicular distance between bases) = ah

Lateral Surface Area (not inc. its bases) of Right Cylinders (inc. Right prisms) = (perimeter of base) (perpendicular distance between bases). **For full surface area, add the bases!**

Volume of cone (inc. pyramids) =  $(\frac{1}{3})$  (area of base) (height)

Lateral Surface area (not inc. base) of right circular cone =  $\pi$  (radius) (slant height) =  $\pi r l$ . **For full surface area, add the base!**

Surface area of sphere =  $4\pi r^2$ ; Volume of sphere =  $(\frac{4}{3}) \pi r^3$

Remember to **include the units**, if any. E.G., area might be  $\text{in}^2$ , while volume might be  $\text{in}^3$ .

**Lesson 4:**

This covers construction. Look it over, but we're going to skip construction after this look-over. The ancient Greeks loved geometric construction, and spent lots of time on it. But in my opinion, Descartes' development of Cartesian coordinates (enabling analytical geometry) made these mostly obsolete, and computers have finished the job. Generally, we'll skip any construction problems from now on.

**Lesson 5:**

You should already know how to do these; if not, read it!!! In particular:

$$\sqrt{x} = x^{1/2} ; \sqrt[n]{x} = x^{1/n} ; \sqrt[n]{x^m} = x^{m/n} ; x^a x^b = x^{a+b} ; \frac{x^a}{x^b} = x^{a-b} ; (x^a)^b = x^{ab}$$

In a 3D rectangle:

$$\text{Length of diagonal} = \sqrt{x^2 + y^2 + z^2}$$

$$i = \sqrt{-1}$$

$$i^2 = (\sqrt{-1})^2 = -1$$

$$i^3 = i i^2 = -i$$

$$i^4 = i^2 i^2 = (-1)(-1) = 1$$

Here are some complex number facts; complex numbers are normally written in the form  $a + bi$  (sometimes  $j$  is used instead of  $i$ ). Complex numbers are used in many technical fields.

Beware of signs when multiplying same-exponent terms – one of our simplification rules doesn't work if the terms are negative (even Euler made this mistake). This is one reason we write  $(-1)^{1/2}$  as “ $i$ ” or “ $j$ ”:

$$\sqrt{a}\sqrt{b} = \sqrt{ab} \text{ and } a^n b^n = (ab)^n \text{ presumes } a > 0, b > 0$$

$$\text{Wrong: } \sqrt{-2}\sqrt{-2} = \sqrt{(-2)(-2)} = \sqrt{4} = 2$$

$$\text{Correct: } \sqrt{-2}\sqrt{-2} = (-2)^{1/2}(-2)^{1/2} = (-2)^{1/2+1/2} = (-2)^1 = -2$$

$$\text{Correct: } \sqrt{-2}\sqrt{-2} = \sqrt{2}i\sqrt{2}i = \sqrt{2}\sqrt{2}ii = -2$$

Given 2 similar figures,  $(\text{area1}/\text{area2}) = \text{scale}^2$

**Lesson 6:**

(Fractional equations, radical equations, 3 linear equations) You should already know how to do these...

**Lesson 7:**

Proofs are important, and fundamental to all math today. Much of science of 2500 years ago is discredited, but their proofs are still just as valid. If you have to prove something to a skeptical audience, the strongest evidence available is a mathematical proof (if the premises are correct).

<i>Valid</i>	<i>Invalid</i>
All men are mortal Socrates is a man -----	All men are mortal Socrates is a man -----
Socrates is mortal	All men are Socrates

Don't confuse "contrapositive" with "converse" or "inverse" (these similar words are often confused).  
When  $P \Rightarrow Q$  ("P implies Q"):

- Contrapositive is:  $\sim Q \Rightarrow \sim P$  i.e., "Not Q implies not P". Contrapositive is always true if the original premise is true.
- Converse is:  $Q \Rightarrow P$ . Does *not* always follow from premise (*could* be true).
- Inverse is:  $\sim P \Rightarrow \sim Q$ .

So given premise "Rainy weather implies that I'm carrying an umbrella":

- Contrapositive is: "I'm not carrying an umbrella, which implies that it is not rainy weather". That has to be true, given the premise; if it *were* raining, I *would* be carrying an umbrella, and since I'm not carrying an umbrella, it must not be raining.
- Converse is: "I'm carrying an umbrella, which implies that it is rainy weather". That might not be true, given only the premise; perhaps I carry an umbrella *every* day.
- Inverse is: "It's not rainy weather, (which) implies that I'm not carrying an umbrella." Again, that might not be true, given only the premise; perhaps I carry an umbrella every day.

### **Lesson 8:**

Look at pictures in 8.B (page 62) and 8.C (page 63) – in certain cases, segments are guaranteed to be proportional, and this will be important in many geometry proofs.

### **THIS WEEK:**

Work through lessons 9-12 this week ("Monday – Thursday"). Try to do it yourself first, but call if you need me!

**Do home study test #2** (covers lessons 5-8). This is not for credit, but I want everyone to try doing a "practice test" before we start counting them. Have parent grade initially (identify which ones right/wrong); I intend to do final grading (so I can give partial credit). Drop "question #18" of home study test 2, when you see it; question #18 involves construction (not worth the trouble).

**Note:** If you follow the suggested schedule, next Monday you'll do lesson #13. We'll cover that in very briefly in class today, so that you'll be ready for it when you get to it. Lesson #13 isn't very hard; it's basically "memorize a bunch of facts", and you know many of them already.