# Advanced Math: Introduction to Calculus

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## Mathematics after pre-calculus/advanced math

There are a number of mathematical areas you may choose to study after this course, including:

1. **Discrete mathematics**: the study of mathematical structures that are fundamentally discrete in the sense of not supporting or requiring the notion of continuity. This is fundamental to modern computing, so if you plan to get involved in computing (e.g., computer science, programming, etc.), plan to take a course in discrete mathematics! This includes many sub-fields, including:
   a. **Logic**: the study of reasoning.
   b. **Graph theory**: the study of mathematical structures used to model pairwise relations between objects from a certain collection. Widely used in computing.
2. **Probability and Statistics**: the study of uncertainty. If you will be analyzing datasets (as a scientist, economist, etc.), you'll want this. You'll need to know calculus first, though.
3. **Calculus**: the study of limits, derivatives, integrals, and infinite series (we'll explain those soon).

For many, the next mathematical step is calculus, because (1) calculus is remarkably useful across many fields, and (2) calculus is a prerequisite for many other mathematical areas. Calculus is not that hard if taught well, but unfortunately there are *lots* of bad calculus teachers. Some universities make it very difficult to succeed by having 7:30am classes, professors whose accents cannot be understood, or by putting hundreds of people in a class (making it practically impossible to ask questions).

So here I will try to give a very brief introduction to Calculus, with the goal of giving you the basics. I obviously can't teach a whole course in two short sessions, but I can prepare you, in case you need it.

A common definition of Calculus is the study of four things: limits, derivatives, integrals, and infinite series. Limits are actually a simple mathematical tool that help you analyze the other three; we've actually hinted at them in various places in pre-calculus. So let's start with limits.

## Limits

A fundamental issue in mathematics is that we cannot divide by zero. A fantastic tool that often enables us to get around this problem (and related problems) is the idea of the "limit". A limit describes the behavior of a function as its argument either "gets close" to some value or as it becomes arbitrarily large.

### *Approaching a constant*

Let's consider this function:

$$f(x)=\frac{x-1}{\sqrt{x}-1}$$

We can't calculate f(1) directly, because that would be a division by 0. But we *can* find a limit as x approaches 1. It turns out that as x approaches 1, f(x) approaches the value 2. I'm not going to try to *prove* that, or how to figure that out, but the following table will hopefully illustrate the idea:

| *x* | *f(x)* |
|---|---|
| 0.9 | 1.95 |
| 0.99 | 1.99 |
| 0.999 | 1.999 |
| 1.0 | (division by zero) |
| 1.001 | 2.001 |
| 1.01 | 2.01 |
| 1.1 | 2.1 |

In standard mathematical notation, this relationship would be described this way:

$$\lim_{x \to 1} f(x) = \lim_{x \to 1} \frac{x-1}{\sqrt{x}-1} = 2$$

which means "as x approaches 1, the value of f(x) approaches 2".

More generally, this mathematical expression:

$$\lim_{x \to c} f(x) = L$$

is read as "the limit of f(x), as x approaches c, is L". This expression doesn't say anything about the value f(c) itself; often, you can't evaluate f(c). For example, f(c) might require division by zero. But it just means that you can make f(x) get closer to the value of L, as x gets closer to the value c.

Here's another example: the expression "x/x" can't be calculated at x=0, because you can't divide by 0. But clearly when x=2, x/x = 1, when x=1, x/x = 1, and when x=0.5, x/x = 1. No matter how close you make x=0, as long as x isn't exactly 0, x/x is 1. In traditional mathematical notation for a limit, you could thus say:

$$\lim_{x \to 0} \frac{x}{x} = 1$$

which is read "the limit of x divided by x, as x approaches 0, is 1". In other words, as x approaches the value of 0, x/x approaches 1 (in fact, it's equal to 1 up to that point). Notice that it doesn't matter that we can't directly compute 0/0. Using limits, we don't need to; we just need to be able to express what value some expression is approaching.

## *Approaching infinity*

Limits can show what happens to an expression when x grows towards infinity, instead of showing what happens as x approaches a particular constant. For example, the function $f(x) = \frac{3x}{x+1}$ gets closer and closer to 3 as x gets larger and larger. This can be expressed as:

$$\lim_{x \to \infty} f(x) = \lim_{x \to \infty} \frac{3x}{x+1} = 3$$

which is read “the limit of f(x), as x goes towards positive infinity, is 3”. Here are a few values that may help convince you that this is true (or at least that it’s likely):

| *x* | *f(x)* |
|---|---|
| 1 | 1.5 |
| 10 | 2.727... |
| 100 | 2.970... |
| 1000 | 2.997... |

If you can calculate limits, then you can also calculate many infinite series using a few rules, since an infinite series may “in the limit of an infinite number of numbers” produce a non-infinite answer.

My goal here is not to show how to calculate limits in general, or the rules about limits; a full calculus class will go into that. But hopefully you can see now that limits are not scary; they’ve already been hinted at before, and they are simply a way of describing what value some expression is tending toward.

A full calculus class will show how to deal with limits in a more rigorous way. In particular, if you think you know the limit value (e.g., by successive approximation), there’s a technique to determine whether or not it’s true. Of course, to use that approach, you need to be able to guess a likely value for it. The limits of a number of expressions are already known, and there are also some general rules for finding limits in many cases. A full class will cover this.

## Derivatives

A derivative tells you the (instantaneous) slope of a function at any point.

### *How to find a derivative from first principles*

Imagine that a function is describing a roller coaster’s track height (off-the-ground) as the function moves from left to right, and that you want to know at any point the slope of the coaster. If the track is continuous (it’d better be!), at any point the coaster will be going up by some amount, down by some amount, or level. But the “obvious” way of figuring out the slope at a particular point will hit a “divide by zero” problem; let’s see why.

You can easily figure out the slope of any function between x=1 and x=2, since their respective heights will be f(1) and f(2):

$$slope = \frac{difference\ of\ y}{difference\ of\ x} = \frac{f(2) - f(1)}{2 - 1} = f(2) - f(1)$$

But that isn’t necessarily the slope at x=1; it’s the slope between two different points where x=1 and x=2. Below is the graph of $f(x)=x^2+1$, along with the line connecting f(1) and f(2). Note that its slope is different than the slope of the tangent at f(1):

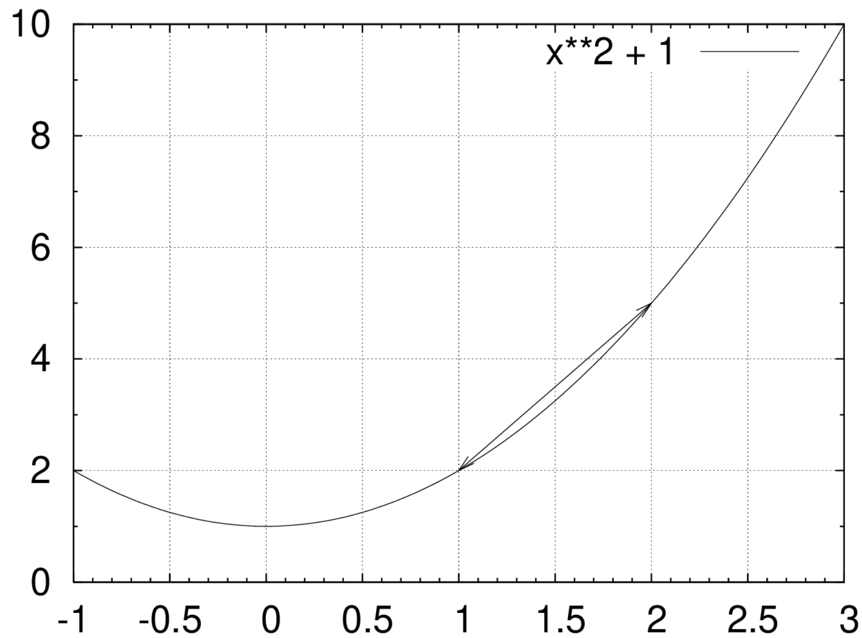


*Illustration 1: Parabola with line from x=1 to x=2*

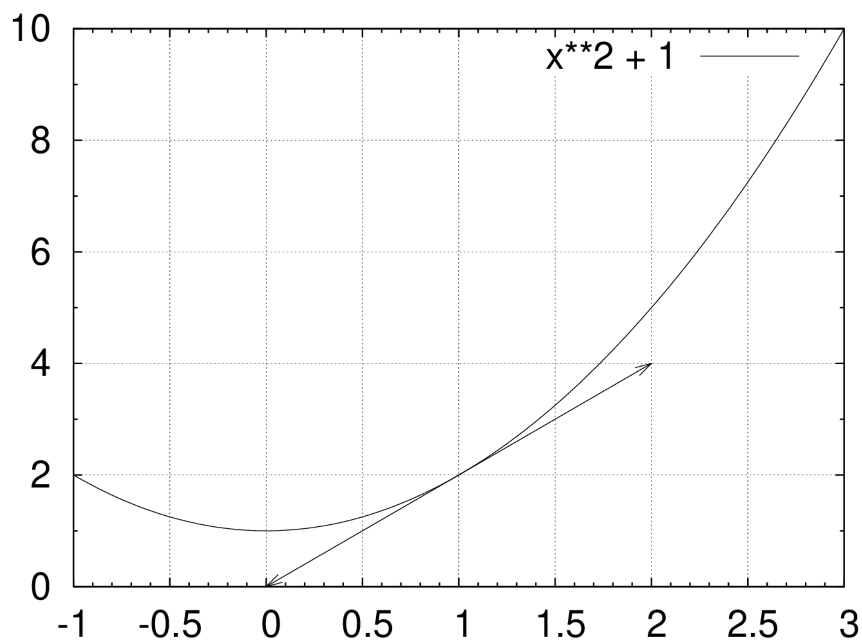


*Illustration 2: Parabola with tangent line at x=1*

We can get a closer approximation of the slope at x=1 by choosing a second point that's closer to x=1, say, x=1.01:

$$slope = \frac{difference\ of\ y}{difference\ of\ x} = \frac{f(2) - f(1)}{1.01 - 1} = \frac{f(2) - f(1)}{0.01}$$

But that's still an approximation. Can we do better? The problem is that we *cannot* simply choose the same point twice, say, f(1) and f(1). If we do, we'll divide by zero:

$$slope = \frac{difference\ of\ y}{difference\ of\ x} = \frac{f(1)-f(1)}{1-1} = \frac{f(1)-f(1)}{0} \text{ Can't divide by zero!}$$

But now that we have a "limit", we can use limits to avoid the problem entirely. What we need is some value "h", which is the distance on the x-axis between the two points, and find the limit as h goes to zero:

$$slope = \frac{difference\ of\ y}{difference\ of\ x} = \lim_{h\to 0} \frac{f(1+h)-f(1)}{(1+h)-1} = \lim_{h\to 0} \frac{f(1+h)-f(1)}{h}$$

More generally, the slope of any function f(x) at x is:

$$slope\ of\ f(x)\ at\ x = f'(x) = \frac{df}{dx} = \frac{difference\ of\ y}{difference\ of\ x} = \lim_{h\to 0} \frac{f(x+h)-f(x)}{(x+h)-x} = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$

Notice that a derivative is itself a function. That is, a "derivative" is a *new function* that tells you the slope of an *original function* at any point.

## Using the definition to find the derivative of $3x^2$

So let's use this definition to find the derivative of $f(x)=3x^2$. First, let's start with the definition:

$$slope\ of\ f(x)\ at\ x = f'(x) = \frac{df}{dx} = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$

Now let's plug in f(x):

$$\frac{df}{dx} = \lim_{h\to 0} \frac{3(x+h)^2-3x^2}{h} = \lim_{h\to 0} \frac{3x^2+6xh+3h^2-3x^2}{h} = \lim_{h\to 0} \frac{6xh+3h^2}{h} = \lim_{h\to 0} \frac{h(6x+3h)}{h}$$

But we've already established that the limit of h/h, as h goes to zero, is 1. Hopefully you can also see that the limit of 3h, as h goes to zero, is 0. (I'm intentionally being a little vague here; a real Calculus course would go through this more rigorously. But I want you to get the *idea* at this point.) That means that:

$$... = \lim_{h\to 0} (6x+3h) = 6x$$

**We have produced a *new* function, 6x, from our original function $f(x)=3x^2$. The new function (which gives the instantaneous slope of the original function at any point) is the *derivative* of the original function, and is typically notated as f ′ (x) or as** $\frac{df}{dx}$ .

So this means that given the function $f(x)=3x^2$, f ′ (x) = 6x. How can we use the derivative? Well, remember that the derivative gives us the *slope* of the first function. So the slope of f(x), at x=4, is 6(4)=24. The slope at x=1 is 6(1)=6, at x=0 is 6(0)=0, and at x=-1 the slope is 6(-1)=-6. In general, for any function f(x), its slope for any value of x is the value of f ′ (x).

## Using the definition to find the derivative of a second-order polynomial

Now let's try $f(x)=ax^2+bx+c$. What is its derivative, f ′ (x)? We can use the general definition to figure this one out, too:

$$f'(x)=\frac{df}{dx}=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}=\lim_{h\to 0}\frac{(a(x+h)^2+b(x+h)+c)-(ax^2+bx+c)}{h}$$

$$=\lim_{h\to 0}\frac{ax^2+2\text{axh}+ah^2+bx+bh+c-ax^2-bx-c}{h}=\lim_{h\to 0}\frac{2\text{axh}+ah^2+bh}{h}=\lim_{h\to 0}\frac{h(2\text{ax}+ah+b)}{h}$$

Again h/h tends toward 1 as h goes to 0, and ah goes to 0 as h goes to 0, leaving:

f ′ (x) = 2ax + b

## *Derivative of a constant*

Doing everything from first principles (like we did above) would be really painful. A good part of calculus is spent learning rules so that you don't have to do everything starting from first principles. For example, if you have a constant c, then the derivative is always 0. Here's a graph of f(x)=3, which may help explain why:

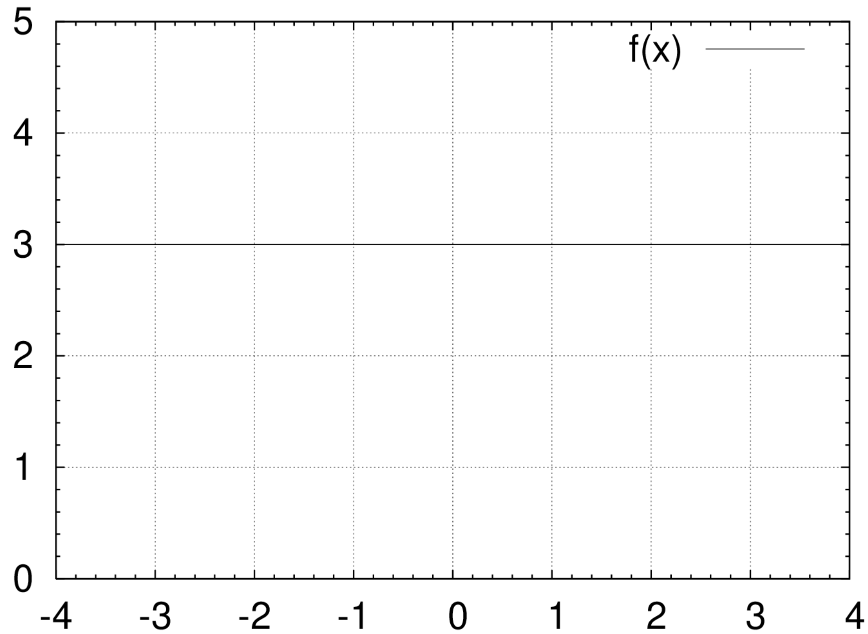



*Illustration 3: Function f(x)=3; a line with slope=0*

If f(x)=c, and c is a constant, then no matter what the value of x is, the result is always c. That's a straight line with slope 0 everywhere, and since derivatives give you the slope, you'll get 0.

## *Derivative of the equation of a line*

Here's another simple example: if you have f(x)=mx+b, then the derivative f ′ (x)=m. That's because if you have an equation of a line, it has the same slope m everywhere.

## *Basic Rules: Adding, Subtracting, Multiplying, Dividing*

Here are some simple rules for when you want to find the derivative of two functions that are added, subtracted, multiplied, or divided by each other:

$$(f(x)+g(x))' = f'(x)+g'(x)$$
$$(f(x)-g(x))' = f'(x)-g'(x)$$
$$(f(x)\cdot g(x))' = f'(x)\cdot g(x)+f(x)\cdot g'(x)$$
$$\left(\frac{f(x)}{g(x)}\right)' = \frac{(f'(x)g(x)-f(x)g'(x))}{g^2(x)}$$

Thus, (3 + 2)' = (3)' + (2)' = 0 + 0 = 0.

If you have some function q(x)=3 · 2, you can find q ′ (x) this way:

q'(x) = (3 · 2)' = (3)'(2) + (3)(2)' = 0(2) + (3)0 = 0

You can use the multiplication rule to find the derivative of 3x. Remember that the derivative of "x" is 1, because the expression f(x)=x is just a line with slope 1:

(3 · x)' = (3)'(x) + (3)(x)' = 0x + 3(1) = 3

You can even use the multiplication rule to find the derivative of $x^2$:

$(x^2)' = (x \cdot x)' = (x)'(x) + (x)(x)' = (1)(x) + (x)(1) = 2x$

The derivative of 3/x is:

$$\left(\frac{3}{x}\right)' = \frac{(3)'(x)-(3)(x)'}{(x)^2} = \frac{0(x)-(3)(1)}{x^2} = \frac{-3}{x^2}$$

## *Derivative of simple exponents*

If you have an expression of the form:

$cx^n$

Where c and n are real numbers, then its derivative is:

$cnx^{n-1}$

So the derivative of $5x^4$ is $20x^3$. You can replace "x" with another simple variable name, but you can't use this rule directly if "x" is a more complicated expression; you'll need something called the "chain rule" to handle more complicated expressions. But first, let's talk about polynomials.

## *Derivative of arbitrary polynomials*

There's a nifty trick to finding the derivative of polynomials. Given any polynomial of this form:

$ax^n + bx^{n-1} + .... + px + q$

Its derivative has this form:

$nax^{n-1} + (n-1)bx^{n-2} + .... + p$

The "q" in the original function disappears in the derivative. That's because the slope of a constant value is zero, and adding zero to something (including the derivative) doesn't change its value. So if:

$f(x) = 2x^5 + 7x^4 -2x^3 -40x +10$

then its derivative is:

$f'(x) = 10x^4 + 28x^3 -6x^2 -40$

More examples:

if $f(x)=2x^{50}-4x^2+2x-5$, then $f'(x) = 100x^{49} - 8x + 2$.

if $f(x)=100x^{49} - 8x + 2$, then $f'(x) = 4900x^{48} - 8$.

if $f(x)=4x^7-2x^3+4x-8$, then $f'(x) = 28x^6 - 6x^2 + 4$.

if $f(x)=-2x^5-2x^2$, then $f'(x) = - 10x^4 - 4x$.

if $f(x)=3x+2$, then $f'(x) = 3$.

if $f(x)=4$, then $f'(x) = 0$.

## *Chain rule (Skip for later)*

The "chain rule" lets find the derivative for very complicated functions just by using a few existing rules. I will not be asking anyone to use the "chain rule" in the bonus quiz; using it well takes practice. But I wanted to briefly mention that it exists, because it's very powerful – it lets you divide complicated problems into smaller problems that you can solve.

To use the chain rule, you take your existing function, split it into two parts, and then use the "chain rule" to find the the derivative using those two parts.

First, the official definition. The chain rule says that:

$$\text{if } h(x)=f(g(x)), \text{then } h'(x)=f'(g(x))\cdot g'(x)$$

So much gibberish, right? Well, here's an example.

Let's find the derivative of $h(x)=(x^3 + 2)^{50}$: Multiplying this out would take a very long time! We know the rule for exponents, but the problem is that the exponent rule only works if "what's inside" is a single variable. So let's split this into two functions, $f(u)=u^{50}$ and $g(x)=x^3+2$. Is this the same thing as h(x)? Let's see: $h(x)=f(g(x))=f(x^3+2) = (x^3 + 2)^{50}$. so these two functions f(x) and g(x), when combined as f(g(x)), is the same as our original h(x).

We then find the derivative of each one; $f'(u)=50u^{49}$, and $g'(x)$ is $3x^2$. Notice that we can now find f'(u), because it only has a single variable that gets an exponent (as required). Now we can recombine them again, using the chain rule:

$$h'(x)=((x^3+2)^{50})'=f'(g(x))\cdot g'(x)=50\,g(x)^{49}\cdot 3x^2=50(x^3+2)^{49}\cdot 3x^2=150\,x^2(x^3+2)^{49}$$

Taking an h(x) and breaking it cleanly into two functions f(x) and g(x) can be an art. The main thing to know is that there's a general rule, called the "chain rule", that lets you break a complicated function into two simpler functions.

## Sine and Cosine (Skip for later)

The derivatives for the sine and cosine are simple, but like the exponent rule require single variables:

$$\frac{d}{dx}(\sin x)=\cos x$$

$$\frac{d}{dx}(\cos x)=-\sin x$$

You can use this, plus the multiplication and division rules above, to find many other derivatives. For example, you can find the derivative of tan x... just remember that tan x = (sin x) / (cos x), and then apply the rule for division:

Because $\left(\frac{f(x)}{g(x)}\right)' = \frac{(f'(x)g(x) - f(x)g'(x))}{g^2(x)}$, therefore

$$\left(\frac{\sin x}{\cos x}\right)' = \frac{(\cos x)(\cos x) - (\sin x)(-\sin x)}{\cos^2 x} = \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x} = sec^2 x$$

Similarly, you can find the derivative of $\sin^2$ x, once you expand that to (sin x)(sin x) and apply the rule for multiplication:

Because $(f(x) \cdot g(x))' = f'(x) \cdot g(x) + f(x) \cdot g'(x)$, therefore

$$((\sin x)(\sin x))' = (\cos x)(\sin x) + (\sin x)(\cos x) = 2 \sin x \cos x$$

If you have something more complicated, you generally must use the chain rule, which I won't emphasize given our short time. Here's an example in case you're curious. To find the derivative of sin $(2x^5)$, you'll need to split it into two functions. We can do this by setting f(x) = sin x, and g(x) = $2x^5$, so that h(x)=f(g(x)) = f($2x^5$) = sin ($2x^5$). We can find the differentials easily; f'(u)=cos u and g'(x)= $10x^4$. Thus:

$$h'(x) = (\sin(2x^5))' = f'(g(x)) \cdot g'(x) = \cos(g(x)) \cdot 10x^4 = \cos(2x^5) \dot{1}0x^4 = 10x^4 \cos(2x^5)$$

## *Importance of derivatives*

Derivatives are really important because they let us describe rate of change. In the real world, things constantly change; derivatives let us describe and analyze those changes, and even predict their results.

For example, "velocity" is the change of a distance over time - which means that velocity is really the derivative of a function of an object's position. You can even repeat this - "acceleration" is the change of velocity over time, so acceleration is actually the derivative of the velocity. In other words, acceleration is really the derivative of a derivative!

## Integrals

As noted above, the derivative gives you the slope (aka the slope of the tangent), at any point, of some function. You can think of the derivative as describing the slope of a roller coaster at any point, if the roller coaster's track is defined by a function.

An *integral* lets you determine the area underneath the roller coaster tracks (between the function position and the line y=0). For example, if you wanted to paint a fence underneath the roller coaster tracks, you could use the integral to find its area (which determines the amount of paint you'd need).

The usual mathematical notation for a "definite" integral from x=a to x=b of function f(x) is this:

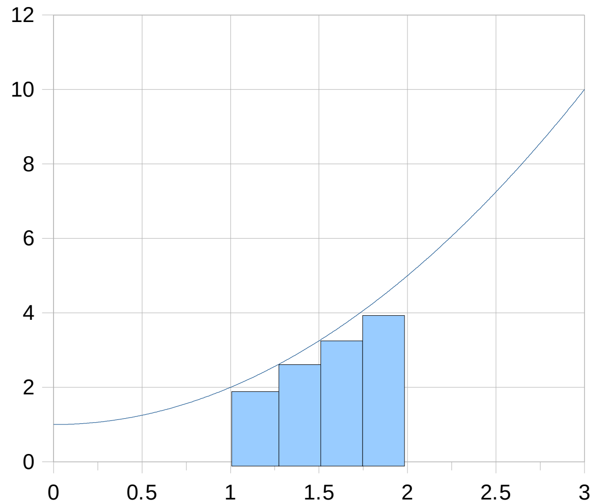
$$\int_a^b f(x)\,dx$$

This is pronounced "the definite integral of f-of-x, with respect to x, from a to b".

If f(x) is always zero or more, then the value of this integral is the area between f(x) (above), y=0 (below), and the lines x=a and x=b. Interesting things happen when f(x)<0; we'll get to that later.

## *Using limits to find integrals*

One way to figure out an approximate area for some function is to break the surface into rectangles, and then add up the areas of the rectangles. We know the area of a rectangle is just the base times the height! So, let's divide the area from x=a to x=b into rectangles, each of which are k width. Let's set the left edge of each rectangle to the value of the function, and then add up the rectangles. For example, here's a curve from x=1 to x=2, divided into 4 rectangles:




The left edge of a rectangle beginning at x will have height f(x), and since all the rectangles have width k, the area of a rectangle beginning at x will be k times f(x). We will have (b-a)/k rectangles; notice that as k gets smaller, we will have more rectangles. The equation that adds up those rectangles is:

$$\text{Approximate area} = \sum_{x=a}^{b} k \cdot f(x) = \left( \sum_{x=a}^{b} f(x) \right) \cdot k$$

Obviously, adding up four rectangles will only approximate the actual area under a curve like this example. But if we make the rectangles narrower and narrower (adding up additional rectangles), we will get more an more accurate. The limit of the sum of these rectangles, as k goes to zero, will be the actual area under the curve - which is the integral.

$$\lim_{k \to 0} \left( \left( \sum_{x=a}^{b} f(x) \right) \cdot k \right) = \int_a^b f(x) dx$$

You can then find actual integrals for various values, but it's a pain to figure out integrals from first principles. Instead, there are much easier ways to find integrals, which we'll do after introducing a new term: the anti-derivative.

## *Anti-derivatives*

Let's say that you have some function F(x), and its derivative f(x) = F'(x). Since the derivative of F(x) is f(x), we can also say that an anti-derivative of f(x) is F(x). This is just going the other way; the "anti-derivative" is the opposite of the derivative. In other words, an anti-derivative of f(x) is any function F(x) that, if you take its derivative, produces f(x).

Why the "any function" stuff? Let's imagine that we have $f(x)=12x^2$. What is its anti-derivative F(x)?

- $4x^3 + 2$ is an anti-derivative of f(x); because by the polynomial rule, $(4x^3 + 2)' = 12x^2$.
- $4x^3 + 7$ is an anti-derivative of f(x); because by the polynomial rule, $(4x^3 + 7)' = 12x^2$.

In fact, you can add any constant to a function, and its derivative won't change. That's because the derivative of a constant is 0. So, when we report an anti-derivative, you *must* add a "+ C" factor to the end of it, to stand for "any constant". In fact, this is a calculus "secret handshake" - if you can remember the "and a constant" at the end of an anti-derivative, you'll sound like you know a lot (because many people keep forgetting it).

So given $f(x)=12x^2$, its anti-derivative $F(x)=4x^3 + C$, because $(4x^3 + C)' = 12x^2$. Similarly, given $f(x)= 8x^3$, its anti-derivative $F(x)=2x^4 + C$, because $(2x^4 + C)' = 8x^3$.

There's a trick for polynomials in general; given:

$$f(x)=a_n x^n+a_{n-1}x^{n-1}+\ldots+a_1 x^1+a_0$$

the anti-derivative is:

$$F(x)=\frac{a_n}{n+1}x^{n+1}+\frac{a_{n-1}}{n}x^n+\ldots+\frac{a_1}{2}x^2+a_0 x+C$$

So given $f(x)=2x^6 + 13x^5 - 2x + 22$, its anti-derivative is:

F(x) = $\frac{2}{7}x^7+\frac{13}{6}x^6-x^2+22x+C$

## *Derivatives and integrals are inverses/Indefinite Integral*

Why introduce the anti-derivative? Because another name for the "anti-derivative" is "indefinite integral". That's right, the anti-derivative *is* an integral! It turns out that derivatives and integrals are inverses of each other, in the same way that multiplication and division are inverses of each other; you can use each to undo the other. And this fact makes it *much* easier to find integrals; the rectangle method above is really painful to do symbolically for even simple cases.

You write an "indefinite integral" by omitting the "from" and "to" values, so it ends up like this:

$$\int f(x)\,dx=F(x)+C \qquad if \quad f(x)=\frac{d}{dx}F(x)$$

This is read, "the indefinite integral of f(x) with respect to x is F(x) plus a constant, if f(x) is the derivative of F(x) with respect to x".

So let's restate the polynomial rule – now we can find the anti-derivative (aka the indefinite integral) of any polynomial, simply because we can find the derivative of a polynomial:

$$\int a_n x^n+a_{n-1}x^{n-1}+\ldots+a_1 x^1+a_0\,dx \quad = \quad \frac{a_n}{n+1}x^{n+1}+\frac{a_{n-1}}{n}x^n+\ldots+\frac{a_1}{2}x^2+a_0 x+C$$

It's easy in practice; just raise the exponent value by one, and divide the coefficient by that new number. Given ... + $12x^5$ + ..., its indefinite integral will include ... + $(12/6)x^6$ + … = + $2x^6$ + ...

Here are a few examples:

$$\int 3x^2\,dx = x^3 + C$$
$$\int 24x^5\,dx = 4x^6 + C$$
$$\int x^3\,dx = \frac{1}{4}x^4 + C$$
$$\int x\,dx = \frac{1}{2}x^2 + C$$
$$\int 3\,dx = 3x + C$$
$$\int 0\,dx = 0x + C = C$$
$$\int 15x^2 + 4\,dx = 5x^3 + 4x + C$$
$$\int 45x^4 - 18x^2\,dx = 9x^5 - 6x^3 + C$$
$$\int 80x^3 - 21x^2 + 24x - 9 = 20x^4 - 7x^3 + 12x^2 - 9x + C$$

## *Finding Definite Integrals using Indefinite Integrals*

If F(x) is an anti-derivative (indefinite integral) of continuous function f(x), then this is always true:

$$\int_a^b f(x)\,dx = F(b) - F(a) \quad \text{given} \quad F(x) = \int f(x)\,dx$$

An integral that goes from some x=a to x=b is called a "definite" integral; you can calculate it if you can find the function's anti-derivative (aka indefinite integral).

Remember that if $f(x)= 8x^3$, its anti-derivative $F(x)=2x^4 + C$, because $(2x^4 + C)' = 8x^3$. You can now use that fact to find its integral from 1 to 4 (which is the area under $8x^3$ from x=1 to x=4):

$$\int_1^4 8x^3\,dx = F(4) - F(1) = (2(4)^4 + C) - (2(1)^4 + C) = 2(256) - 2 = 510$$

Notice that the constant Cs cancel out; that *always* happens with definite integrals. So as long as you're computing a definite (not indefinite) integral, you can drop the C's (I'll do that from now on).

Here's another example for calculating definite integrals; let's say $f(x)=x^5+3$. We can calculate its definite integral from -2 to +2 as follows:

$$\int_{-2}^{+2} x^5 + 3\,dx = F(2) - F(-2).\ \text{Since } F(x) = \frac{1}{6}x^6 + 3x + C \text{ and C's always cancel in definite integrals,}$$
$$= \left(\frac{1}{6}(2)^6 + 3(2)\right) - \left(\frac{1}{6}(-2)^6 + 3(-2)\right) = \left(\frac{64}{6} + 6\right) - \left(\frac{64}{6} + -6\right) = \frac{64}{6} + 6 - \frac{64}{6} + 6 = 12$$

Let's do another example. If $f(x)=20x^4 - 14x$, then its anti-derivative $F(x)= 4x^5 - 7x^2 + C$. To find the area under f(x) from x=10 to x=12, you compute this definite integral (note that the C's are dropped):

$$\int_{10}^{12} 20x^4 - 14x\,dx = F(12) - F(10) = (4(12)^5 - 7(12)^2) - (4(10)^5 - 7(10)^2) = 595020$$

Let's find the area of f(x) from x=2 to x=4, when $f(x) = 12x^3 - 6x^2 + 7$. First, let's figure out the anti-derivative; per the process above, this is $F(x) = 3x^4 - 2x^3 + 7x + C$. Always double-check your anti-derivative by going the other way (in this case it's fine). Now, let's find the area from x=2 to x=4:

$$\int_2^4 12x^3 - 6x^2 + 7 = F(4) - F(2) = (3(4)^4 - 2(4)^3 + 7(4)) - (3(2)^4 - 2(2)^3 + 7(2)) = 668 - 46 = 622$$

### *Integrals of sums and differences*

Functions that are added or subtracted can be easily integrated; the rule is basically the same as for derivatives:

$$\int_a^b f(x)+g(x)\,dx=\int_a^b f(x)\,dx+\int_a^b g(x)\,dx$$

$$\int_a^b f(x)-g(x)\,dx=\int_a^b f(x)\,dx-\int_a^b g(x)\,dx$$

### *Real World: Negative areas and Volumes*

I've been careful to give examples where f(x)>0 in the areas to be calculated. What happens when f(x)<0? Well, then we end up computing "negative areas". The real world, however, doesn't have "negative areas", so if you're using integrals to compute "real world areas" you need to find out where f(x)<0, and if it is, compensate. I won't go into the details here, I just want to make you aware of this.

You don't always need to compensate; there are other reasons to compute integrals where negative values are fine. If you're just given a math problem as an integral, then just compute it (and don't worry if they are negative or not).

A volume is really an "area of an area", so integrals can help find volumes too. Volumes are basically "integrals of integrals".

### *There's more*

There's more, but the purpose here is to give just a brief taste, so you'll have an idea of what Calculus is all about. A real course will cover important rules (such as l'Hôpital's rule) that make it easy to handle a wide range of circumstances. Calculus is widely studied *because* it's useful in an extraordinary number of circumstances. I hope that this introduction will give you a "leg up" if you ever take Calculus.