Advanced Math: Notes on Lessons 114-117

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Lesson 114: Graphs of Factored Polynomial Functions

This lesson explains how to quickly graph polynomials, particularly higher-powered ones; it also helps you get a "feel" for what these polynomials do. The book does a good job on this, so read it; here we'll just note the highlights.

A *turning point* is "hump" in the graph where it changes direction (it changes from going up to going down, or changes from going down to going up). A polynomial always has fewer turning points than its degree, e.g., a fifth-order polynomial has at most 4 turning points.

The *sign of the highest-degree term* determines the polynomial's eventual direction as it goes towards positive and negative infinity. The eventual direction when going toward negative infinity is also affected by whether that largest exponent is even or odd, e.g., x^2 as x goes towards negative infinity is positive, but x^3 as it goes towards negative infinity is negative. This is all because if you multiply an even number of negative numbers, the result is negative; if you multiply an odd number of negative numbers, the result is positive. If the exponent is positive, then sign of the coefficient is the sign of the polynomial as it tends to both positive and negative infinity. Here's a table that summarizes this:

Highest-degree term			***Resulting polynomial value***	
Coefficient	***Exponent***	***Example***	***Towards +∞***	***Towards -∞***
+	even	$5x^2$	+	+
-	even	$-5x^2$	-	-
+	odd	$5x^3$	+	-
-	odd	$-5x^3$	-	+

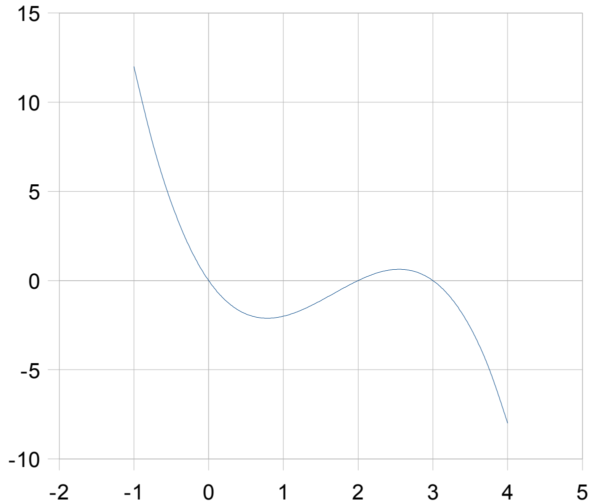
Finally, there are often *zeros* (that is, values for x where the polynomial produces 0). You can find these trivially if the polynomial is factored; in (x-2)(x+3), x=2 and x=-3 are zeros.

So for example, if you have f(x) = x(x-2)(3-x), you can quickly figure out its general shape. First, note that this will become a 3rd-degree polynomial, i.e., $(x^2-2x)(3-x) = -x^3 + 5x^2 -6x$. Then note:

- Turning points: This is 3rd-degree, so it has at *most* 2 turning points
- Sign of highest-degree: This is -, and the coefficient is odd, so it will go negative as x goes to positive infinity, and it will go positive as x goes to negative infinity
- Zeros: The polynomial evaluates to 0 at x=0, x=2, and x=3 (see the factoring above).

There's something else of interest - flex points. If you look at a curve, some curves could "hold water" (this is called "concave up"), while others would only hold water if they were upside down ("concave down"). Places where it switches between concave up and concave down are called "flex points". See the graphs in the book for more.

Now you can try a few additional x values (in particular, ones between the zeros) to see what it does. Here's what you should come up with:



Lesson 116 will continue with some useful tips.

This lesson introduces a mathematical fact that is far more general: if pq=0, then p is 0, q is 0, or both are 0. This is called the "zero factor theorem".

Lesson 115: The Remainder Theorem

This lesson is about an interesting oddity that you *may* find useful in evaluating polynomials: It turns out that you can use polynomial division to evaluate the value of a polynomial!

Let's start with an example. Let's say we have the polynomial $f(x)=x^3+2x^2-x+1$, and divide it by x-3. We will get a new polynomial and the remainder 43:

$$\frac{x^3+2x^2-x+1}{x-3}=x^2+5x+14+\frac{43}{x-3} \quad \text{because} \quad \begin{array}{r|rrrr} 3 & 1 & 2 & -1 & 1 \\ & \downarrow & 3 & 15 & 42 \\ \hline & 1 & 5 & 14 & 43 \end{array}$$

Oddly enough, if we compute f(3), that will *also* be 43:

$$f(x)=x^3+2x^2-x+1$$
$$f(3)=3^3+2(3)^2-(3)+1=27+18-3+1=43$$

More generally, if you divide a polynomial f(x) by x-c, the remainder is the same as f(c); this is the remainder theorem.

Here's why this happens - imagine that we have two polynomials, P(x) and Q(x), where Q(x) is the result of dividing P(x) by x-c and throwing away the remainder:

$$\frac{P(x)}{x-c}=Q(x)+\frac{r}{x-c} \qquad \textit{Given}$$
$$P(x)=Q(x)(x-c)+r \qquad \textit{Multiply by x-c}$$
$$P(c)=Q(c)(c-c)+r \qquad \textit{This is what happens if x=c}$$
$$P(c)=r \qquad \textit{c-c is zero, zero times anything is zero}$$

(Strictly speaking, this proof is invalid because you can't divide by zero. That can be easily solved by simply accepting the second step as your starting point, which would be equivalent... but then you might not see where it came from.)

Lesson 116: Region of Interest

(This is a continuation of lesson 114.)

There's a simple trick for finding the "region of interest" for polynomials - that is, where interesting things like the roots, turning points, and flex points are:

1. Divide the whole polynomial by the coefficient of the highest-degree term; the result is the normalized polynomial equation.
2. Find the largest coefficient ignoring the sign (i.e., the absolute value), and add one to it; this is the radius of the region of interest.

The "region of interest" is inside the circle whose center is the origin, and whose radius is the radius of the region of interest. All roots (real and complex) are within the circle; in addition, the x values of all the turning points and flex points are within the circle.

For example, given:

$$2x^3 + 4x^2 - 6x + 4$$

We first create the normalized polynomial... in this case by dividing by 2 (the coefficient of x^3):

$$x^3 + 2x^2 - 3x + 2$$

The largest coefficient, ignoring the sign, is 3. Thus a circle centered at the origin with radius 3+1=4 will contain all the polynomial roots, as well as the x values of its turning points and flex points.

Lesson 117: Prime and Relatively Prime Numbers / Rational Roots Theorem

Prime and Relatively Prime

This section emphasizes definitions that you probably already know.

Composite number: A counting number(1, 2,...) that can be expressed as the product of two other counting numbers both greater than 1. E.G., 12 is composite because 12=2x2x3.

Prime number: A counting number that is not composite. E.G., 7 is prime because the only counting number product that produces 7 is 1x7.

Relatively prime: Counting numbers whose only common factor is 1. E.G., 25 and 27 are relatively prime, even though neither one is a prime number (5x5 and 3x3x3).

Fundamental theorem of arithmetic: Every time a counting number is written as a product of prime factors, the same factors must be used.

Rational Roots Theorem

Roots of polynomial f(x) are the value of x where f(x)=0; there's an algorithm to find all rational roots.

First, two useful facts to know:

1. Every polynomial of degree n has exactly n roots (though the roots may duplicate).
2. If a real polynomial (all coefficients are real) has a+bi as a root, then a-bi is also a root.

A shortcut to root-finding is the *rational root theorem,* which says that if you have a polynomial equation of the form $ax^n + ... +z = 0$, where a is the coefficient of the highest-degree term, z is the constant, and ***all*** coefficients (including z) are integers, then all the rational roots (if any) have form:

$$\textit{Rational roots} = \pm \frac{\textit{factors of constant } z}{\textit{factors of coefficient } a}$$

So, you can simply create a list of all the possible rational roots, and then calculate the polynomial for each value to see if it produces 0... if it does, then it's a root. Note: If the polynomial equation includes non-integers, just multiply both sides by some number to *make* all the coefficients integers.

Of course, this doesn't tell you if there *are* rational roots; some or all may be irrational or complex. But it can help, and this works with any power. The quadratic equation can find the roots for any polynomial of degree 2, but it won't help beyond 2. What's more, once you find the rational roots, that can help you factor the polynomial into something smaller, which may be the key to the rest. E.G., if "5" is a root, then the whole polynomial = (x-5)(something else); that means you can divide the equation by x-5 and see what's left to factor.

So for $12x^8+5x^2+2=0$, using this approach, we can list the possible rational roots as:

$$\frac{\pm\{1,2\}}{\{1,2,3,4,6,12\}} = \pm 1, \pm\frac{1}{2}, \pm\frac{1}{3}, \pm\frac{1}{4}, \pm\frac{1}{6}, \pm\frac{1}{12}, \pm 2, (\textit{skipping duplicates}) \pm\frac{2}{3}$$

The "skipping duplicates" is where combinations are skipped because they are duplicate values we've already listed. E.G., 2/2 is a possible combination, but 2/2 equals 1 and we already listed that. To see if any are an *actual* root, insert them into the original polynomial and see if they evaluate to zero. Use the remainder theorem (lesson 115, above) to do that quickly, and you can quickly find rational roots!

Aside: The general approaches to polynomials in this lesson and others (such as114 and 116) exist because as polynomial degrees get larger, exact solutions are *much* harder. The quadratic equation is a single equation which solves polynomials of degree 2. In 1824, Niels Henrik Abel proved that there can be *no* general formula (involving only the arithmetical operations and radicals) for the roots of a polynomial of degree 5 or greater in terms of its coefficients (this is the "Abel-Ruffini theorem"). There *are* equations that solve polynomials for degrees 3 and 4, and they've been known for hundreds of years, but they are *much* more complicated. I do not expect you to memorize those – they're the kind of thing you look up if you need them. For example, here's the general equation for degree 3:

$$f(x) = ax^3+bx^2+cx+d$$

You can factor it into $a(x-x_1)(x-x_2)(x-x_3)$ by finding the roots x_1, x_2, and x_3 this way:

$$\text{Let } q = \frac{3ac - b^2}{9a^2} \text{ and } r = \frac{9abc - 27a^2 d - 2b^3}{54a^3}$$

$$\text{Let } s = \sqrt[3]{r + \sqrt{(q^3 + r^2)}} \text{ and } t = \sqrt[3]{r - \sqrt{(q^3 + r^2)}}$$

$$\text{Then } x_1 = s + t - \frac{b}{3a},\ x_2 = -\frac{1}{2}(s+t) - \frac{b}{3a} + \frac{\sqrt{3}}{2}(s-t)i,\ x_3 = -\frac{1}{2}(s+t) - \frac{b}{3a} - \frac{\sqrt{3}}{2}(s-t)i$$

Again, I do *not* expect you to memorize this solution for degree 3!! I show this only to show that things get complicated quickly as the degrees get larger. That is why general approaches, that work with polynomials of arbitrary degree, are needed.