Advanced Math: Notes on Lessons 122-125

David A. Wheeler, 2009-05-01

Lesson 122: Graphs of Rational Functions / Graphs that Contain Holes

A "rational function" is a function that can be written as a polynomial divided by a polynomial:

$$f(x)=\frac{a_n x^n+...+a_1 x^1+a_0}{b_m x^m+...+b_1 x^1+b_0}$$

Hopefully, you can see that the zeros of the numerator are zeros of the entire function. The zeros of the denominator are called the "poles" of the function; the poles are where the values go towards positive or negative infinity. If you want to graph it, it's very useful to know where the zeros and poles are! Any polynomial f(x) is also a rational function, because you can write it as "f(x)/1", and 1 is a (really trivial) polynomial.

If we add a restriction that both polynomials must have only real roots, and no duplicate roots, then we can factor both the top and bottom into some constant "c" multiplied by factors of the form (x - a) where each "a" is a (different) root. With this restriction, at every zero the function *must* cross between positive and negative values. With this additional information, it's not hard to sketch the graphs of the function; the book shows how.

Graphs that contain holes

Polynomials are continuous; they don't contain "holes". That is, for any value of x, you can always get a value for f(x) if it's a polynomial.

That is *not* true for many other functions, for example, a rational function can contain "holes" where certain values of x are not allowed (i.e., holes in the domain). For example, this rational function:

$$f(x)=\frac{x^2-6\mathrm{x}+8}{x-2}$$

Can be factored as:

$$f(x)=\frac{(x-2)(x-4)}{x-2}$$

Obviously, this expression can't have a value at x=2 because that would involve division by zero. Can we cancel the "x-2" factors? Yes, but canceling is still only allowed when the bottom is nonzero, so we end up with this:

$$f(x)=\frac{(x-2)(x-4)}{x-2}=x-4 \quad (x\neq 2)$$

Graphing "x-4" is easy (it's just a line), but at x=2, you'll need to draw a little empty circle to indicate "this value of x is not allowed".

Lesson 123: The General Conic Equation

A conic section is a curve that can be formed by intersecting a right circular conical surface with a plane. All conic sections can be described by this equation:

$$ax^2 + bxy + cy^2 + dx + ey + f = 0$$

By choosing different coefficient values you can make this a parabola, circle, ellipse, a hyperbola, or a "degenerate conic section". A "degenerate conic sections" is one of the following: a point, a line, two parallel or intersecting lines, or absolutely no points at all.

Shape	***Condition (in all cases b=0)***	***Resulting Equation***
Parabola	$a\neq0$, $e\neq0$, $c = 0$	$ax^2 + ey + f = 0$ ► $y = (-a/e)x^2 + (-f/e)$
Circle	$a=c$, $a\neq0$	$ax^2 + ay^2 + dx + ey + f = 0$ ► $(x - x_c)^2 + (y - y_c)^2 = r^2$
Ellipse	$a\neq c$, ($a>0$ and $c>0$) or ($a<0$ and $c<0$)	$ax^2 + cy^2 + dx + ey + f = 0$ ► $(x - h)^2/m^2 + (y - k)^2/n^2 = 1$
Hyperbola	($a>0$ and $c<0$) or ($a<0$ and $c>0$)	$ax^2 + cy^2 + dx + ey + f = 0$ ► $(x - h)^2/m^2 - (y - k)^2/n^2 = 1$

If $b\neq0$, then we have an "xy" term, which rotates the shape. The book will not cover rotation in general, but this lesson does cover a special case, ones which simplify to:

$$xy = \text{constant}$$

This special case is a hyperbola rotated 45°; these are hyperbolas occupying either (a) the first and third quadrant, or (b) the second and fourth quadrant.

Lesson 124: Point of Division Formulas

On a number line, to move from one coordinate to another you add the "coordinate change", which is just the "final - initial". This is true for x and y coordinates as well.

If you want to move a *fraction* of a coordinate, you multiply the coordinate change by the fraction. So, if you have an initial point (x_1, y_1), and you want to move "towards" a final point (x_2, y_2) by "n/d" amount, the coordinate changes would be:

$$\Delta x = \frac{n}{d}(x_2 - x_1)$$

$$\Delta y = \frac{n}{d}(y_2 - y_1)$$

Given an initial point, to move it this way, you'd replace each x with "x+Δx" and y with "y+Δy". You'd do the same with equations, too; replacing the x's and y's this way will move it by that amount.

Example: "Find the equation for point F's coordinates that lies 2/7 of the way between point P1 (x_1, y_1) and point P2 (x_2, y_2)":

$$\begin{aligned} new\ x &= x_1 + \Delta x & new\ y &= y_1 + \Delta y \\ &= x_1 + \frac{2}{7}(x_2 - x_1) & &= y_1 + \frac{2}{7}(y_2 - y_1) \\ &= x_1 + \frac{2}{7}x_2 - \frac{2}{7}x_1 & &= y_1 + \frac{2}{7}y_2 - \frac{2}{7}y_1 \\ &= \frac{5x_1 + 2x_2}{7} & &= \frac{5y_1 + 2y_2}{7} \end{aligned}$$

Lesson 125: Using the Graphing Calculator

One challenge with describing how to use a graphing calculator is that there are many models, each of which work a little differently. They often have different limitations, too (e.g., the TI-82 can't graph x^c if c is a non-integer fraction and $x<0$). Generally, all can graph one or more functions, solve systems of equations, or find roots.

Graphing: To graph a function, you need to first enter the function, and then command it to create a graph. The window settings matter, of course; typically you can set the minimum and maximum of both X and Y, and then zoom in/out or pan in any direction.

Solving simultaneous equations: To solve simultaneous equations, you can enter the various functions, and then look for their intersection. Graphing calculators usually have some sort of "intersect" function; typically you'd enter a point that's "close to" the intersection.

Roots: To find a root, generally you enter the equation without the "= 0" part, graph it, and then ask it to find a root. You will usually need to identify the range to look in (the minimum and maximum x), or some sort of "starting position".

This is probably a good place to talk about computer programs/tools for math. There are many, many other computer programs/tools that can help perform mathematical tasks; graphing calculators are only one example of such tools. Some are "open source"; anyone can review, modify, and redistribute such programs, and thus they end up being developed using processes similar to how mathematics itself is developed (they also tend to be free, which is nice!). Here are some general categories:

- Spreadsheets. These present you with a grid of cells, and in each cell you can type in an expression to be calculated (which may depend on other cells). They typically come with hundreds of predefined functions. A common one is Microsoft Excel (part of Microsoft Office). A good alternative that is free & open source software is Calc (part of OpenOffice.org, which I used to write these notes; I find OpenOffice.org to be a better word processor.)
- Computer Algebra Systems. These allow you to manipulate symbols and can do many algebraic tasks, and many can also do plotting and numerical work too. Common for-pay ones are Mathematica, Maple, and MathCAD. Good freely-available ones (all open source) are Maxima/wxMaxima; I hear good things about Yacas and the Java Algebra System (JAS).
- Numerical systems. These are languages or computer libraries focused on doing complicated

numerical calculations (often ones that would take too long or be inaccurate without very special measures). Well-known ones include Octave and LAPACK (both open source).

- Statistical packages. These are designed to do statistical analysis, and can typically handle "complicated" or "marginal" cases not well-handled by others. A common system is "R" (open source).
- Word processors that can edit/display mathematical formulas. Most major word processors can edit and display traditional math notation. There's often big difference between what is finally displayed on paper and what you edit. That's because most mathematicians find it easier to edit math formulas using a "flat" text layout which is then turned into a pretty display on the screen. There *are* programs that let you edit math notations directly, but it often gets hard to "select" the right component or make some kinds of modifications, which is why this approach tends to be used instead. In OpenOffice.org, for example, you would "Insert/Object/Formula", type in:

  ```
  x={-b +- sqrt {b^2 - 4ac}} over 2a
  ```

  and the program will display:

  $$x=\frac{-b\pm\sqrt{b^2-4\mathrm{ac}}}{2\mathrm{a}}$$

- LaTeX (open source software). It's pronounced "lah – tech" or "lay-tech" (I use the first one). LaTeX is a powerful "typesetting system" used for producing scientific and mathematical documents with high typographic quality. To use it, you type your information into a plain text file, and it then figures out how to format it using sophisticated typesetting algorithms. It's very widely used in the technical publishing industry for academic journals, particularly by mathematicians, physicists and other people who have complex notational requirements. Elsevier, IEEE, the Royal Society, and George Mason University's dissertation templates all provide author guidelines for people who use LaTeX.
- Web browsers/MathML. If you want to post nontrivial formulas to the web, the standard format for doing so is MathML (which is often used along with HTML). Some browsers (like Firefox) can automatically display MathML; others require a plug-in to handle them.

These categories aren't really that separate, and I expect systems to increasingly become more capable and "borrow" capabilities from each other and/or work together. But most tools have a specialty (something they're especially good at), and you may have trouble trying to use a tool in ways it wasn't designed for.

Often these programs are designed to have a separate "front end" user interface and a "back end" that does the calculations; in some cases, you get and install them separately. For example, "Maxima" is a computer algebra system back-end; most users will install "wxMaxima" which proviedes a graphical front-end to Maxima. Some programs are front-ends to multiple tools, for example, SAGE is a user interface that integrates and lets you control many different "back end" mathematical tools.

In most mathematical tools, you need to mark multiplication in some way, usually with "*". So instead of writing "ab", you'd write "a*b". Matrix multiplication may be denoted separately (e.g., Maxima uses the dot "." instead). Division is typically notated as "/".

And with that, we're done with the book.