Advanced Math: Notes on Lessons 21-24
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If something in Saxon is absolutely confusing, feel free to consult another book. A good short one is Bob Miller's "PreCalc for the Clueless"; it doesn't cover as much as Saxon, but his book is pretty clear on what it does cover. You can get it (and similar books) at your local public library.

May want to re-examine lesson 13D, products of the *entire* secant segments (inc. the external portion) is equal to the product of the the *external* secant segments.

Lesson 21: Evaluating Functions / Domain and Range / Types of Functions / Tests for Functions
This lesson is primarily about defining terminology and notation, particularly on functions. This is *really* important; a vast amount of math is built on top of functions. Know what domain, range, 1-to-1, etc. are. A function must pass the "vertical line" test; a 1-to-1 function must *also* pass the "horizontal line" test.

Saxon is unfortunately ambiguous when introducing functions. He says that functions are "single-valued" (correct), but what he means is that a function only *produces* a single value. Lesson 21.D is intended to clarify this, but you might easily miss that. For a function, a vertical line must hit at *most* one number (if it produces real numbers). Later on, you'll see that you *can* create functions that *accept* multiple inputs (the set of those inputs is the function's domain, so even then a function only has *one* domain), but they still produce only one input (there are tricks to overcome that limitation too, but that's more advanced). So it's the "only produce one output" rule that's important for functions.

The basic idea is that you can define a function by just saying "f(x)=...", and when you have a particular value for x, just replace every instance of "x" on the right-hand-side with that particular value. **Be careful**: I *highly* recommend that your right-hand replacements have (..) around them to avoid a common beginner's mistake.

So, if $f(x)=x^2+2x+3$, then:
$f(5) = (5)^2+2(5)+3 = 25+10+3 = 38$
$f(2y) = (2y)^2+2(2y)+3 = 4y^2+4y+3$
$f(\theta) = (\theta)^2 + 2(\theta) + 3 = \theta^2 + 2\theta + 3$ *Note that the free variable can be changed.*
WRONG: $f(y+1) = (y+1)^2 + 2y + 1 + 3 = y^2 + 2y + 1 + 2y + 4 = y^2 + 4y + 5$
RIGHT: $f(y+1) = (y+1)^2 + 2(y + 1) + 3 = y^2 + 2y + 1 + 2y + 2 + 3 = y^2 + 4y + 6$

Lesson 22: Absolute Value / Reciprocal Functions
Saxon shows graphical solutions to these absolute-value function questions, using a few patterns. If they help, great! In particular, if you can reorganize your equation into one of these patterns, you can create a corresponding graph:

| x – center | <= bound "Everything from center–bound to center+bound, inclusive"
| x – center | >= bound "Everything <= center–bound, and everything >= center+bound"

If it's < instead of <=, or > instead of >=, then it's not inclusive (put a hole at the edge).

If you have more complex equations, or you find that approach confusing, here's an alternative approach that is completely symbolic/algebraic, it works all the time, and always works the same way.

First, note that a more traditional way of representing the “absolute” function is this:

$$\text{abs}(x)=|x|=\begin{cases} x \; if \; x\geq 0, \\ -x \; if \; x<0 \end{cases}$$

So, you can solve any “absolute value function” problem by splitting the problem into two parts (≥0 and <0), solving each part, and then merging the parts once you’ve completely solved each part. E.G.:

$$|x+2|\geq 3$$
$$\rightarrow\begin{cases} (x+2)\geq 3 & if\,(x+2)\geq 0, \\ -(x+2)\geq 3 & if\,(x+2)<0 \end{cases}$$
$$\rightarrow\begin{cases} x\geq 1 & if\; x\geq -2, \\ x+2\leq -3 & if\; x<-2 \quad \text{Notice the flip in direction} \end{cases}$$
$$\rightarrow\begin{cases} x\geq 1 & if\; x\geq -2, \\ x\leq -5 & if\; x<-2 \end{cases}$$
$$\rightarrow x\geq 1 \text{ or } x\leq -5 \quad \text{Notice how to merge them back}$$

Note that the “if” here is essentially an “and”, that is, *both* the statements before and after the “if” have to be true. In contrast, either or both of the two separate lines can be true. So:

$$\begin{cases} (x+2)\geq 3 & if\,(x+2)\geq 0, \\ -(x+2)\geq 3 & if\,(x+2)<0 \end{cases}$$

has essentially the same meaning as:

$$\begin{cases} (x+2)\geq 3 & \text{and}\,(x+2)\geq 0, \\ & \text{OR} \\ -(x+2)\geq 3 & \text{and}\,(x+2)<0 \end{cases}$$

Normally you can simplify each of the “X and Y” (“X if Y”) expressions:

- If they are both “<”, e.g., “x < c1 and x < c2”, the smaller value (choosing between c1 and c2) wins, so it simplifies to simply “x < c” where c is c1 if that’s the smaller value, otherwise c2. Think about it; if “x” has to less than both c1 and c2, then clearly it has to be smaller than whichever value is smallest. Remember that signs matter, e.g., -5 is smaller than 2. So “x < -5 and x < 2” simplifies to “x < -5”.
- If they are both “>”, e.g., “x > c1 and x > c2”, then the larger value (between c1 and c2) wins. So it will simplify to “x > c”, where c is c1 if that’s the larger value, otherwise it’s c2.
- If they differ in direction, they will often end up defining a range. Flip one of them so that they “go the same way”, and then solve. Thus, you can solve “x< 5 and x > 2” by flipping “x > 2” into “2 < x”, and then combine them into “2 < x < 5”. Sometimes you’ll find that it’s not possible, e.g., “x < 0 and x > 2”; which simplifies to “no solution”.

Beware. When you have an inequality (<, <=, >, >=), and you multiply or divide by a negative number, the inequality *flips* (i.e., < becomes >, ≤ becomes ≥, and so on). Strictly speaking, this happens with “=” too, but the “flip” of “=” is “=” so it doesn’t matter.

Lesson 23: Exponential Function / Sketching Exponential

The main thing here is that certain types of functions have a characteristic “look”. Functions of the form $f(x)=b^x$ where x>1 have a characteristic rapid growth, which you should recognize on sight.

Unfortunately, exponential equations can describe the amount of time it takes a computer to do some really useful problems (where x is a measure of the size of the problem, and f(x) is the time it takes). In computing there's a special field called "complexity theory", which can describe generally how long a program takes to run when varying its input size. If a program will be exponential, you need to make sure it only deals with really small problem sizes, or find another approach. *Knowing* that this is the case is the first step to solving it.

Lesson 24: Sums of Trigonometric Functions / Combining Functions
Part 24A is really just practice creating trig values of standard angles, and then simplifying.

Lesson 24B can be head-spinning if you let it. It's important, but it's also a simple idea, don't get fooled into thinking this is more complex than it is. I wish Saxon had used simpler examples, because with simpler examples it becomes really obvious what's going on. So here are my simple examples; given:

$f(x) = x + 1$, and $g(x) = 2x$

Then:

$(f+g)(x) = f(x)+g(x) = (x+1) + (2x) = 3x+1$
$(f-g)(x) = f(x) -g(x) = (x+1) - (2x) = -x+1$
$(f\cdot g)(x) = f(x) \cdot g(x) = (x+1) \cdot (2x) = 2x^2+2x$
$(f/g)(x) = f(x) / g(x) = (x+1) / (2x) = 1/2+1/(2x) \quad (x\neq 0)$

The sneaky new operation is "composition", symbolized by "○". It just means apply the rightmost function, then use its result as the input to the next function. So continuing our example:

$(f\circ g)(x) = f(g(x)) = f(2x) = (2x) +1 = 2x+1$

Notice that the order matters, just like order matters for subtraction and division:

$(g\circ f)(x) = g(f(x)) = g(x+1) = 2(x+1) = 2x+2$

There are computer programming languages (notably Haskell) built on functional composition.

THIS WEEK: Work through lessons 21-24 this week ("Monday – Thursday").

Do home study test #5 this week (covers lessons 17-20), presumably on Friday. Have a calculator, but few of the problems will be useful with one (and some have "do not use a calculator" instructions). Please review your notes before taking it, and memorize what you need first. *Show your work*, so can give partial credit. Have a parent grade it initially (to identify which ones are right or wrong), and put it in the new mailbox on Sunday morning. I intend to do the final grading, so that I can give partial credit (or full credit if it's just an equivalent expression).