Advanced Math: Notes on Lessons 37-40
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Lesson 37: Locus, midpoint

The midpoint is just the point "between" two other points. Thus, its x and y coordinates are just the average of the other points' x and y coordinates.

Lesson 38: Fundamental Counting Principle and Permutations / Designated Roots / Overall Average Rate

A permutation = arrangement in a particular order, without repetition.
The letters A, B, C can be arranged in 6 different ways, that is, 6 different permutations. That's because in the first position you can choose any of the 3 letters; the next one can only be one of the 2 letters left; and the last position is always the one letter left. So that's 3x2x1. Note... that's a *factorial*. So if you have n different letters, the number of permutations is n!. [Walkthrough in class if have time: Languages have Subject (S), Verb (V), and Object (O), but different ones have different "default" orders. How many orders of these three are possible? Answer: 3x2x1 = 6.]

If repetition *is* permitted, then each position has the same number of possibilities, the result is different. You still multiply the "number of different options" at each position together, but they don't decrease, so if there are n different symbols, and you have x different positions, the number of sets is n^x. [Walkthrough]

The rest is straightforward.

Lesson 39: Radian measure of angles / forms of linear equations

In math, we normally measure angles in *radians*, not *degrees* – so much so that, if its units aren't marked, you should assume angles are in radians. What's a radian? If you sweep out an angle until the length of the arc is equal to the length of the circle, that angle is one radian (~57.3°). This is exactly true:
2π radian = 360° (degrees).

Here are examples of converting radians to degrees and back:

$$1\,rad \times \left(\frac{360°}{2\pi\,rad}\right) \approx 57.3°$$

$$90° \times \left(\frac{2\pi\,rad}{360°}\right) = \frac{\pi}{2}\,rad$$

Note that *length of arc* = (*central angle in radians*) (*length of radius*)

Usually when we work with lines we'll use the "slope-intercept" form, e.g.:
$y = mx + b$
This is handy because it's actually a function, that is, $y = f(x) = mx+b$.
But in a few cases, you sometimes need to use the "general form" of the line equation, which is:
$Ax + By + C = 0$
This is more general because it can easily represent horizontal and vertical lines. You can't reasonably represent a vertical line in y=mx+b.

Lesson 40: The argument in mathematics / The laws of logarithms / Properties of inverse functions

The "arguments" of a function are simply the inputs into the function. Currently you've only seen function notation used with one argument, but functions can have more than one argument. In fact, you've actually already seen functions with more than one argument. For example, adding is really a function that takes two arguments (often called the "left" and "right") and returns a single value.

Remember that:

$$10^2 \cdot 10^3 = 10^{2+3} = 10^5$$

If you take the log (base 10) of all sides, you get:

$$\log_{10}(10^2 \cdot 10^3) = \log_{10} 10^{2+3} = \log_{10} 10^5$$

which simplifies to:

$$\log_{10}(10^2 \cdot 10^3) = 2 + 3 = 5$$

Something funny is going on here! Basically, if you use them in a clever way, logarithms can "convert" multiplication into addition; they also turn division into subtraction, and turn exponentiation into multiplication. This is why slide rules were historically so useful; rulers can only "add" and "subtract" lengths, but by clever marking they turn adding and subtraction into multiplication and division. This is actually a common trick in math; if some operation is hard, we may be able to convert it into some other operation that's easy. Here are the key laws you'll need to memorize and use:

$$\log_b(x \cdot y) = \log_b x + \log_b y$$

$$\log_b \frac{x}{y} = \log_b x - \log_b y$$

$$\log_b x^y = y \log_b x$$

You can call the last law "pop goes the weasel"; the exponent (y) can "pop" out of the logarithm argument entirely! So you can compute $\log_{10} 500 + \log_{10} 2 = \log_{10}(500*2) = \log_{10}(1000) = 3$.

Finally, a basic definition: two functions f(x) and g(x) are *inverses* of each other if:
f(g(x)) = x, for all x in the domain of g, and
g(f(x)) = x, for all x in the domain of f.

The book shows a more complex example; let's start simpler (since hopefully that will make it clearer). Let's say that f(x)=x+1, and g(x)=x-1. Just with your eyes it should be obvious that these are inverses of each other, but let's prove it:
f(g(x)) = f(x-1) = (x-1)+1 = x **YES**
g(f(x)) = g(x+1) = (x+1)-1 = x **YES**

Getting 100% on the tests

All of you understand the material, and I really want all of you to be getting 100% on the tests. Unfortunately, all of you are making a number of "silly mistakes" that are preventing that. Don't accept that – strive for perfect scores! In particular, look over your past tests – don't just see what the error was, but try to figure out *why* you made that mistake, and *what could you change* to reduce the likelihood of ever making *that* kind of mistake again. Here are some ideas that may help:

1. Re-read the question before answering, and copy it carefully.

2. Pretend that you're writing on a board to explain it to someone, not just writing it for yourself. (You *are* writing it for someone – the grader!). So label various parts, and show how they connect together. Pretend that you're making an argument to a skeptical jury, who will only convict if you convince them that each step is absolutely correct.
3. Do less in each step. Many of your are combining multiple steps into a single written step. If they're still clear, and you can do that without error, great. But if you are making errors, then it's obviously too much. Slow down, and do less in each step. If you take 5 steps to do something accurately, it's better than taking only 3 steps if you sometimes make a mistake that way.
4. Write larger and clearly. If it's hard for others to read, it's probably hard for you to read too, and you may copy something wrongly. Line up each step, so it's easier to see when you accidentally drop a denominator or an exponent. Use more paper, it's cheap.
5. Watch out for sign errors. If you're multiplying or dividing, be careful that you get the right final sign. For angles, clockwise is a *negative* angle.
6. If you're computing A-B, where both are polynomials, do that in two steps: make the polynomial B into -B by reversing all the signs, and then add them. This works because A-B = A + (-B). But I find you're less likely to make a mistake because you've now split the work into two steps: (1) sign-swapping (easy to do) and (2) adding (which tends to be easier than subtracting, since the order no longer matters). This is an example of an active technique you can use to *counter* sign errors before they can event start.
7. Is there a potential for multiple answers, such as having angles or using positive exponents? E.G., $x^2=4$ has *two* solutions (both 2 and -2), not just one. Make sure *all* answers are included.

When you think you're done, check it somehow:

1. If you solved for an unknown, plug the solution back into the equation and see if it really works.
2. If you solved for a set of values (e.g., an inequality), pick a few "check values" and try them out. Check values on the boundaries themselves; these can particularly catch errors, in particular.
3. Look at the answer – does it make sense? Do you have more than 100% of ingredients? Negative values where only positive makes sense? Outrageously large/small velocities? Does it imply an infinity somewhere that makes no sense in the original?
4. Walk through and check each step. Did you copy everything? Or did something get dropped (some exponent, a denominator, a sign)? This is easier if you label everything and make it easy for someone else to follow; the someone else may be you :-).

And... insert your idea here! All of us have a few types of errors that we tend to make (I make sign errors, too!). Good mathematicians are aware of their weaknesses, and specifically choose to use approaches to counteract them. For example, making errors with the square root of -1 was so common that mathematicians invented a notation (i) to reduce the risk. In fact, the history of math is full of old notations that have been abandoned because it was too easy to make mistakes in them. If there are two ways to do something, use the way in which you're least likely to make an error (or do it both ways, and use them as checks on each other). Look for patterns of errors, and use your creativity to work around your own weaknesses. This kind of self-awareness – being aware of your weaknesses and compensating for them – can help you in many other areas of your life too...!

Do home study test #9 this week (covers lessons 33-36), presumably on Friday. Please review your notes before taking it, and memorize what you need first. *Show your work*, so can give partial credit. Have a parent grade it initially (to identify which ones are right or wrong). I intend to do the final grading, so that I can give partial credit (or full credit if it's just an equivalent expression).