Advanced Math: Notes on Lessons 78-81

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Lesson 78: Hyperbola

The ellipse is the set of all points where the sum of the distances to two points (the foci) is the same.

The hyperbola is something like an "opposite" of an ellipse. A hyperbola is the set of all points where the absolute value of the *difference* of the distance to two points (the foci) is the same. I.E.:

$$| \text{distance}_1 - \text{distance}_2 | = \text{constant}$$

When both foci are on the x-axis, and the center is at the origin, the hyperbola has this standard form:

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

Notice that when y=0, we have two solutions: (-a, 0) and (+a,0). Notice also that this is exactly the same as an ellipse, but with "-" instead of "+".

When both foci are on the y-axis, and the center is at the origin, the hyperbola standard form is:

$$\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$$

Lesson 79: De Moivre's Theorem / Roots of Complex Numbers

Part A: De Moivre's Theorem

If you multiply two complex numbers in polar form, you can get the answer this way:

$$a_1 \cdot a_2 = (r_1 \, cis \, \theta_1) \cdot (r_2 \, cis \, \theta_2) = (r_1 r_2) \, cis \, (\theta_1 + \theta_2)$$

(where r_1 and r_2 are positive.)

You can use this to find exponents. Abraham De Moivre did so, and discovered a pattern, which you'll need to memorize:

$$a^n = (r \, cis \, \theta)^n = r^n \, cis \, (n \theta)$$

Part B: Roots of Complex Numbers

Every complex number except 0 has n *n*th roots - and this includes all real numbers like 8, since 8 is just 8+0i. So there are actually 3 cube roots of 8. Normally, if we give a real number, by convention we expect only the real-number answer (if there is one), and if there are both positive and negative answers, only the positive one. But that's only a convention.

To find all the solutions of a root, first find the solution for one of them in vector form with the smallest angle measure; you can do this by reversing De Moivre's theorem. This means that:

$$\sqrt[n]{a}=\sqrt[n]{r\, cis\, \theta}=\sqrt[n]{r}\, cis(\theta / n) \quad because \quad a^n = r^n\, cis(n\theta)$$

The other solutions all have the same magnitude; just keep adding 360°/n. (The angles of all of the nth roots differ by 360°/n). Remember that for any nth root, you will have n solutions.

Example: Find the 3 cube roots of 8i.

Since 8i = 0+8i = 8cis(90°), the first cube root is:

$$\sqrt[3]{8\, cis\, 90°}=\sqrt[3]{8}\, cis(90°/3)=2\, cis\, 30°$$

To find the other roots, we repeatedly add 360°/3 = 120° to the angle. This means that the three cube roots of 8i are {2 cis 30°, 2 cis 150°, 2 cis 270°}.

We can check our work by using De Moivre's Theorem:

$(2 \text{ cis } 30°)^3 = 2^3 \text{ cis } (3 \cdot 30°) = 8 \text{ cis } (90°)$

$(2 \text{ cis } 150°)^3 = 2^3 \text{ cis } (3 \cdot 150°) = 8 \text{ cis } (450°) = 8 \text{ cis } (90°)$ (subtracting 360° once)

$(2 \text{ cis } 270°)^3 = 2^3 \text{ cis } (3 \cdot 270°) = 8 \text{ cis } (810°) = 8 \text{ cis } (90°)$ (subtracting 360° twice)

Lesson 80: Trigonometric Identities

The key addition in this lesson is a critical trig identity:

$\sin^2 \theta + \cos^2 \theta = 1$

This is actually just another form of the Pythagorean theorem ($a^2 + b^2 = c^2$). Given a right triangle with sides a, b, and c, where θ is the angle opposite a, let's simplify the left-hand-side:

$\sin^2 \theta + \cos^2 \theta$

$(a/c)^2 + (b/c)^2$

$(a^2 + b^2)/c^2$

But from the Pythagorean theorem, we know that:

$a^2 + b^2 = c^2$

So

$(a^2 + b^2)/c^2 = c^2 / c^2 = 1$

which is what was claimed.

Hopefully, you can see that:

$\sin^2 \theta + \cos^2 \theta = 1$

is easily turned into:

$\sin^2 \theta = 1 - \cos^2 \theta$

$\cos^2 \theta = 1 - \sin^2 \theta$

With substitutions, you can also turn this into:

$1 + \cot^2 \theta = \csc^2 \theta$

$\tan^2 \theta + 1 = \sec^2 \theta$

Example 80.2 from Saxon is especially good at showing how you can *use* this; let's go through it:

$$\tan\theta + \frac{1}{\tan\theta} \stackrel{?}{=} sec\,\theta\, csc\,\theta$$

$$\frac{\sin\theta}{\cos\theta} + \frac{\cos\theta}{\sin\theta} = \frac{\sin\theta\sin\theta}{\cos\theta\sin\theta} + \frac{\cos\theta\cos\theta}{\sin\theta\cos\theta} = \frac{\sin^2\theta}{\cos\theta\sin\theta} + \frac{\cos^2\theta}{\cos\theta\sin\theta} = \frac{\sin^2\theta + \cos^2\theta}{\cos\theta\sin\theta} =$$

$$\frac{1}{\cos\theta\sin\theta} = \frac{1}{\cos\theta} \cdot \frac{1}{\sin\theta} = sec\,\theta\, csc\,\theta$$

Here's another example; let's try to determine the constant value of $\tan^2 \theta - \sec^2 \theta$:

$$\tan^2\theta - sec^2\theta = \frac{\sin^2\theta}{\cos^2\theta} - \frac{1}{\cos^2\theta} = \frac{\sin^2\theta - 1}{\cos^2\theta} = \frac{(1-\cos^2\theta)-1}{\cos^2\theta} = \frac{-\cos^2\theta}{\cos^2\theta} = -1$$

Note the substitution of $\sin^2 \theta$ with $1-\cos^2 \theta$. You can check this, e.g., see if this is true for $\theta=45°$:

$$\tan^2 45° - sec^2 45° = (1)^2 - \frac{1}{\cos^2 45°} = 1 - \left(\frac{2}{\sqrt{2}}\right)^2 = 1 - \left(\frac{4}{2}\right) = -1$$

The check doesn't prove it's *always* true (in fact, the original expression isn't -1 when $\cos \theta = 0$), but checking a constant by trying a specific sample can detect a large number of mistakes.

Lesson 81:Law of Cosines

In a right triangle, $a^2 + b^2 = c^2$. The "law of cosines" generalizes this equation to cases where there might not be a right angle. Given some angle P and its opposite length p, and two other sides a and b:

$$p^2 = a^2 + b^2 - 2ab \cos P$$

The last part is the "correction factor". If P is 90°, then since cos 90° = 0, this equation simplifies down to the Pythagorean theorem $p^2 = a^2 + b^2$.

Saxon lists recommended ways to solve for unknowns of triangles; I'd modify it to this list:

1. If have two angles, and want a third angle, just remember that the angles must total 180°
2. If you have two lengths, want a third length, and it's a right triangle, use $c^2 = a^2 + b^2$
3. If it's a right triangle, use sin/cos/tan.
4. If the values of a pair (angle and opposite side) are known, use law of sines. Warning: Arcsin on a calculator always gives a first/fourth-quadrant answer, which may not be what you need - be careful.
5. If values of a pair are not known, use the law of cosines. Law of cosines doesn't have the ambiguity problem of the law of sines