# Advanced Math: Notes on Lessons 86-89

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## Lesson 86: Arithmetic Progressions / Arithmetic Mean

This introduces new terminology:

- sequence: A group of terms (such as numbers) in a definite order
- finite sequence: A sequence with a finite number of elements
- infinite sequence: A sequence with an infinite number of elements
- progression: A sequence where each term depends on the previous one
- arithmetic progression: A progression where each term equals the previous term plus some constant (the "common difference"), i.e., $t_n = t_{n-1}$ + constant

In an arithmetic progression:

First term = $t_1$

Second term = $t_1 + k$ (where "k" is the common difference)

Third term = $t_1 + 2k$

Fourth term = $t_1 + 3k$

$n^{th}$ term = $t_1 + (n-1)k$

## Lesson 87: Sum and Difference Identities / Tangent Identities

### *Sum and Difference Trig Identities*

You need to memorize the following trig identities (you should already know the last two):

sin (A+B) = sin A cos B + cos A sin B

sin (A-B) = sin A cos B - cos A sin B

cos (A+B) = cos A cos B - sin A sin B

cos (A-B) = cos A cos B + sin A sin B

$$\tan A = \frac{\sin A}{\cos A}$$

$\sin^2 A + \cos^2 A = 1$ (this is the Pythagorean theorem; see lesson 80)

Example: Simplify cos(x+π/4)

$$\cos x \cos(\pi/4) - \sin x \sin(\pi/4) \Rightarrow \cos x \frac{1}{\sqrt{2}} - \sin x \frac{1}{\sqrt{2}} \Rightarrow \frac{\cos x - \sin x}{\sqrt{2}} \Rightarrow \frac{\sqrt{2}(\cos x - \sin x)}{2}$$

## *Tangent identities*

tan x = (sin x) / (cos x), and "x" does not need to be a single variable. Using the preceding identities for sin and cos, we can derive these identities:

$$\tan(x+y)=\frac{\tan x+\tan y}{1-\tan x\tan y}$$

$$\tan(x-y)=\frac{\tan x-\tan y}{1+\tan x\tan y}$$

The book shows how to do this; the "trick" is to divide the left-hand-side of the denominator by itself (to get 1), which means you have to divide everything else by that number... which simplifies the result. Let's go through this, because it's important (as a result, and as an example of trig manipulations):

$$\tan(x+y)\Rightarrow\frac{\sin(x+y)}{\cos(x+y)}\Rightarrow\frac{\sin x\cos y+\cos x\sin y}{\cos x\cos y-\sin x\sin y}\Rightarrow\frac{\sin x\cos y\frac{1}{\cos x\cos y}+\cos x\sin y\frac{1}{\cos x\cos y}}{\cos x\cos y\frac{1}{\cos x\cos y}-\sin x\sin y\frac{1}{\cos x\cos y}}$$

$$\Rightarrow\frac{\frac{\sin x}{\cos x}+\frac{\sin y}{\cos y}}{1-\frac{\sin x\sin y}{\cos x\cos y}}\Rightarrow\frac{\tan x+\tan y}{1-\tan x\tan y}$$

## Lesson 88: Exponential functions (growth and decay)

The following form of equation happens often, including in economics (growth of money through compound interest), biology (population growth), electronics (decay of voltage in a capacitor), and physics (radioactive decay). It's sometimes called the "simple exponential growth" function:

$$A_t = A_0\, e^{kt}$$

Where $A_0$ is called the "initial value" or "initial amount", and is the value when t=0.

When k>0, the values increase quickly; when k<0, they decrease.

When time is given and you're solving for amounts, this is easy. E.G., if you assume simple exponential growth for some germ, a starting amount of 5,000, and k=0.5/hour, the amount 8 hours later is:

$$A_8 = (5000)e^{(0.5)(8)} = \sim 272900$$

But if you need to solve for time (t) or the growth/decay constant (k), you need to apply logarithms. This transforms the equation as shown:

$$\ln A_t = \ln(A_0\, e^{kt})$$

$$\ln A_t = \ln A_0 + \ln e^{kt}$$

$$\ln A_t = \ln A_0 + kt$$

## Lesson 89: The Ellipse (2)

In lesson 71 we covered some basics about ellipses.

Here's a quick recap: An ellipse is a "squished circle". Instead of a single center, it has two points (singular: focus; plural: foci). An ellipse is the set of points that have the same (distance to focus #1) + (distance to focus #2).

An ellipse has a "major axis" (that goes through the two foci), and a "minor axis" (perpendicular to the major axis, and going through the midpoint between the foci). Conventionally "a" is 1/2 the length of the major axis, and "b" is 1/2 the length of the minor axis.

If the ellipse's major axis is horizontal, the standard form is:

$$\frac{x^2}{a^2}+\frac{y^2}{b^2} = 1$$

If the ellipse's major axis is vertical, the standard form is:

$$\frac{x^2}{b^2}+\frac{y^2}{a^2} = 1$$

But how can you find or use the foci locations in such equations? First, we need to realize that the foci are somewhere along the major axis. If the foci were on top of each other, we'd have a circle instead of an ellipse. As soon as the foci start moving apart, the circle will "flatten", extending the circle along the line connecting the foci (thus becoming the major axis), and squashing the perpendicular line (the minor axis). So, the foci are somewhere along the major axis – but where?

Well, the distance from the center of the ellipse to an edge along the major axis is "a" by definition. That means that the total distance from one point on the major axis to the other point on the major axis is simply "2a", since an ellipse is symmetrical.

An ellipse is the loci of all points that have the same value of "distance to focus1 + distance to focus2"; what is that value? If we pick an edge that intersects the major axis, and use our yarn, we can see that there's a line segment from the edge to the nearest focus that's duplicated twice, and the distance of the other focus to the other edge isn't covered at all. By symmetry these two segments have equal length, so the total "distance to focus1 + distance to focus2" turns out to just be the length of the major axis inside the ellipse - "2a"!

Now pick an edge that intersects the minor axis. The total distance must be "2a", and here by symmetry the distance to each focus must be identical – thus, the distance to either focus is "a". We know that the length from the origin to that point is b (by definition), and we have a right triangle here... so we can use the Pythagorean theorem. **But be careful**; here "a" and "b" are the ellipse lengths, and *not* their conventional Pythagorean meaning. In particular, the *hypotenuse* is "a", not "c". For clarity, let's use "f" as the distance from the center to a focus; since "a" is the hypotenuse, we have:

$$a^2 = f^2 + b^2 \quad \text{aka} \quad f^2 = a^2 - b^2$$

Again, "f" is the distance from the origin to either foci.