Advanced Math: Notes on Lessons 90-93

David A. Wheeler, 2011-02-21

Lesson 90: Double-angle/Half-angle identities

This lesson does double-duty; it shows identities for sin 2A, cos 2A, sin A/2, and cos A/2... and also shows how these kinds of expressions can be manipulated to simplify an expression.

Since:

$$\sin (A + B) = \sin A \cos B + \cos A \sin B$$

We can find sin 2A by replacing B=A:

$$\sin 2A = \sin A \cos A + \cos A \sin A = 2 \sin A \cos A$$

Similarly, since:

$$\cos (A + B) = \cos A \cos B - \sin A \sin B$$

We can determine that:

$$\cos 2A = \cos A \cos A - \sin A \sin A = \cos^2 A - \sin^2 A$$

And since $\sin^2 A + \cos^2 A = 1$, these are true too:

$$\cos 2A = \cos^2 A - \sin^2 A = (1-\sin^2 A) - \sin^2 A = 1 - 2 \sin^2 A$$

$$\cos 2A = \cos^2 A - \sin^2 A = \cos^2 A - (1 - \cos^2 A) = 2\cos^2 A - 1$$

We can use the last two equations to find half-angle identities – what sin (x/2) and cos (x/2) are.

Let's start with the first one:

$$\cos 2A = 1 - 2 \sin^2 A$$

$$\sin^2 A = \frac{1}{2}(1 - \cos 2A)$$

$$\sin A = \pm\sqrt{\frac{1 - \cos 2A}{2}}$$

$$\sin \frac{x}{2} = \pm\sqrt{\frac{1 - \cos x}{2}}$$

The last step replaces "A" with "x/2"; you replace all occurrences of an arbitrary expression with another, as long as you don't violate math rules (e.g., dividing by zero).

$$\cos 2A = 2\cos^2 A - 1$$
$$\cos^2 A = \frac{1}{2}(\cos 2A + 1)$$
$$\cos A = \pm\sqrt{\frac{\cos 2A + 1}{2}}$$
$$\cos\frac{x}{2} = \pm\sqrt{\frac{\cos x + 1}{2}}$$

Lesson 91: Geometric Progressions

Lesson 86 introduced the "progressions" (a sequence where each term depends on the previous one) and the "arithmetic progression" (a progression where each term equals the previous term plus some constant, the "common difference").

If you multiply the previous term, instead of adding, you have a geometric progression, and the constant is then called the "common ratio", often shown as "r".

So if the common ratio=2, and the first term is 3, we have the geometric progression:

3, 6, 12, 24, ...

The "geometric means" are merely the intermediate values. So if we have have a geometric progression starting at 4, ending at 108, and we want to know the two geometric means between them, that means that we have this progression:

4, ___, ___, 108

So we have 4, $4r^1$, $4r^2$, and $4r^3 = 108$. With the last one we can solve for r:

$r^3 = 108/4 = 27$

$r = 3$

So the real progression is:

4, 12, 36, 108

thus the two geometric means are 12 and 36.

One problem with Saxon's material is that he often doesn't show why anyone would care about these math facts, or how to use them. So let's see a real example! It turns out that the standard Western musical scale ("equal temperament") is built on geometric progression. A note is an "octave higher" when it is twice the frequency of the original pitch. In typical Western music, each note's frequency is determined by multiplying the previous note by a constant value (r). You can determine the value of r just from this information, along with the knowledge of how many notes there are in an octave:

A, A#, B, C, C#, D, D#, E, F, F#, G, G#, A (octave, 2x original)

Since this is a geometric progression, they will have numerical frequencies of these values:

A, _, _, _, _, _, _, _, _, _, _, _, 2A   where "A" is the frequency of the first A

$= Ar^0, Ar^1, Ar^2, Ar^3, Ar^4, Ar^5, Ar^6, Ar^7, Ar^8, Ar^9, Ar^{10}, Ar^{11}, Ar^{12}$

But since $Ar^{12} = 2A$, dividing by "A" on both sides yields:

$$r^{12}=2$$
$$r=\sqrt[12]{2}=1.059463\ldots$$

So if A=440 Hz (modern standard pitch), then the notes immediately above it are:

$$440, 440\left(\sqrt[12]{2}\right)^1, 440\left(\sqrt[12]{2}\right)^2, 440\left(\sqrt[12]{2}\right)^3, 440\left(\sqrt[12]{2}\right)^4, 440\left(\sqrt[12]{2}\right)^5, 440\left(\sqrt[12]{2}\right)^6, \ldots$$

which comes out to about:

$$440, 466.1638, 493.8833, 523.2511, 554.3653, 587.3295, 622.2540, 659.2551, \ldots$$

So the C above A=440Hz is about 523.25Hz.

Lesson 92: Probability of Either (Intersection and Union) / Notations for Permutations & Combination

When two events are mutually exclusive, the probability of either of the two events happening is the sum of their individual probabilities. E.G., a coin flip can be heads or tails, but not both... the probability of a coin flip being *either* heads or tails = 50% + 50% = 100% (presuming it can't land on-edge, etc.).

This can be expressed as follows – given two events X and Y:

$$P(X \text{ or } Y)=P(X \cup Y)=P(X)+P(Y) \quad \text{if X and Y are mutually exclusive}$$

Where there odd-shaped "U" is pronounced "union".

Many events aren't mutually exclusive. E.G., calculating the probability of "drawing a face card (Jack, Queen, or King) or a heart from a 52-card deck" combines two different factors, and it's possible to do both simultaneously. That's because while there are 12 face cards and 13 hearts in a traditional 52-card deck, there are three cards that are both: Jack of hearts, Queen of hearts, and King of hearts. When X and Y aren't mutually exclusive, we need to compensate for this or we'll double-count the event. Thus, in the more general case, the probability of *either* of two events happening (including possibly both happening) is calculated as follows:

$$P(X \text{ or } Y)=P(X \cup Y)=P(X)+P(Y)-P(X \cap Y)$$

Where the last term is:

$$P(X \cap Y)=P(X \text{ intersect } Y)=P(\text{both X and Y occur simultaneously})$$

So the probability of getting a face card or a heart is:

$$P(\text{face} \cup \text{heart})=P(\text{face})+P(\text{heart})-P(\text{face} \cap \text{heart})=\frac{12}{52}+\frac{13}{52}-\frac{3}{52}=\frac{22}{52}=\frac{11}{26}$$

Similarly, the probability of getting a King or a heart from a 52-card deck is:

$$P(\text{King} \cup \text{heart}) = P(\text{King}) + P(\text{heart}) - P(\text{King} \cap \text{heart}) = \frac{4}{52} + \frac{13}{52} - \frac{1}{52} = \frac{16}{52} = \frac{4}{13}$$

Lesson 93: Advanced Trig Identities / Triangle inequalities

Trig equations can be hard to simplify – as I've noted many times before, the individual trig identities are not very powerful, and so it often requires a lot of trickery to change equations into a form where you can use them. Walk through the examples in the book, which should help. If we have time in class, we'll do one.

In general, look at what you have, and think about "what you wish you had instead". Then see if there's a way to multiply, divide, or do something else to get there. Don't worry about having to try several approaches and reaching a dead end; that's pretty normal, and you'll get an intuition about what's most likely to work after trying things out.

A few general tips: The identity $\sin^2 A + \cos^2 A = 1$ is very useful. It means that $\sin^2 A = 1 - \cos^2 A$, and that $\cos^2 A = 1 - \sin^2 A$. Note that you can factor $1 - \cos^2 A = (1 + \cos A)(1 - \cos A)$, too! This means that if you see cosine, and you'd like to combine it with sine, you may be able to convert it into one of these forms. For example, it's often useful to convert things into $\cos^2 A$ or $\sin^2 A$ (so you can use this identity) – typically by multiplying out to get it. You may also find it's useful to divide something by itself to get a "1" - because there are various identities including this one with a "1".

"Triangle inequalities" is actually an extension of the Pythagorean theorem. If in a right triangle:

$c^2 = a^2 + b^2$

Then in an obtuse triangle (angle > 90°):

$c^2 > a^2 + b^2$

While for an acute triangle (angle < 90°):

$c^2 < a^2 + b^2$